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A BANDWIDTH THEOREM FOR APPROXIMATE DECOMPOSITIONS

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Abstract. We provide a degree condition on a regular n-vertex graph G which ensures the existence of a near optimal packing of any family $\mathcal{H}$ of bounded degree n-vertex k-chromatic separable graphs into G. In general, this degree condition is best possible.

Here a graph is separable if it has a sublinear separator whose removal results in a set of components of sublinear size. Equivalently, the separability condition can be replaced by that of having small bandwidth. Thus our result can be viewed as a version of the bandwidth theorem of Böttcher, Schacht and Taraz in the setting of approximate decompositions.

More precisely, let δ_k be the infimum over all $\delta \geq 1/2$ ensuring an approximate K_k-decomposition of any sufficiently large regular n-vertex graph G of degree at least δn. Now suppose that G is an n-vertex graph which is close to r-regular for some $r \geq (\delta_k + o(1))n$ and suppose that $H_1, \dots, H_t$ is a sequence of bounded degree n-vertex k-chromatic separable graphs with $\sum_i e(H_i) \leq (1-o(1))e(G)$. We show that there is an edge-disjoint packing of $H_1, \dots, H_t$ into G.

If the H_i are bipartite, then $r \geq (1/2 + o(1))n$ is sufficient. In particular, this yields an approximate version of the tree packing conjecture in the setting of regular host graphs G of high degree. Similarly, our result implies approximate versions of the Oberwolfach problem, the Alspach problem and the existence of resolvable designs in the setting of regular host graphs of high degree.

1. Introduction

Starting with Dirac's theorem on Hamilton cycles, a successful research direction in extremal combinatorics has been to find appropriate minimum degree conditions on a graph G which guarantee the existence of a copy of a (possibly spanning) graph H as a subgraph. On the other hand, several important questions and results in design theory ask for the existence of a decomposition of K_n into edge-disjoint copies of a (possibly spanning) graph H, or more generally into a suitable family of graphs $H_1, \dots, H_t$.

Here, we combine the two directions: rather than finding just a single spanning graph H in a dense graph G, we seek (approximate) decompositions of a dense regular graph G into edge-disjoint copies of spanning sparse graphs H. A specific instance of this is the recent proof of the Hamilton decomposition conjecture and the 1-factorization conjecture for large n [12]: the former states that for $r \geq \lfloor n/2 \rfloor$, every r-regular n-vertex graph G has a decomposition into Hamilton cycles and at most one perfect matching, the latter provides the corresponding threshold for decompositions into perfect matchings. In this paper, we restrict ourselves to approximate decompositions, but achieve asymptotically best possible results for a much wider class of graphs than matchings and Hamilton cycles.

1.1. Previous results: degree conditions for spanning subgraphs.

Minimum degree conditions for spanning subgraphs have been obtained mainly for (Hamilton) cycles, trees, factors and bounded degree graphs. We now briefly discuss several of these. Recall that Dirac's theorem states that any n-vertex graph G with minimum degree at least $n/2$ contains a Hamilton cycle. More generally, Abbasi's proof [1] of the El-Zahar conjecture determines the minimum degree threshold for the existence of a copy of H in G where H is a spanning union of vertex-disjoint cycles (the threshold turns out to be $\lfloor (n + odd_H)/2 \rfloor$, where odd_H denotes the number of odd cycles in H).

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Komlós, Sárközy and Szemerédi [33] proved a conjecture of Bollobás by showing that a minimum degree degree of $n/2 + o(n)$ guarantees every bounded degree n-vertex tree as a subgraph (this was later strengthened in [35, 13, 26]).

An F*-factor* in a graph G is a set of vertex-disjoint copies of F covering all vertices of G. The Hajnal-Szemerédi theorem [24] implies that the minimum degree threshold for the existence of a K_k-factor is $(1 - 1/k)n$. This was generalised to kth powers of Hamilton cycles by Komlós, Sárközy and Szemerédi [34]. The threshold for arbitrary F-factors was determined by Kühn and Osthus [38], and is given by $(1-c(F))n+O(1)$, where $c(F)$ satisfies $1/\chi(F) \leq c(F) \leq 1/(\chi(F)-1)$ and can be determined explicitly (e.g. $c(C_5) = 2/5$, in accordance with Abbasi's result).

A far-reaching generalisation of the Hajnal-Szemerédi theorem [24] would be provided by the Bollobás-Catlin-Eldridge (BEC) conjecture. This would imply that every n-vertex graph G of minimum degree at least $(1 - 1/(\Delta + 1))n$ contains every n-vertex graph H of maximum degree at most Δ as a subgraph. Partial results include the proof for $\Delta = 3$ and large n by Csaba, Shokoufandeh and Szemerédi [14] and bounds for large Δ by Kaul, Kostochka and Yu [28].

Bollobás and Komlós conjectured that one can improve on the BEC-conjecture for graphs H with a linear structure: any n-vertex graph G with minimum degree at least $(1 - 1/k + o(1))n$ contains a copy of every n-vertex k-chromatic graph H with bounded maximum degree and small bandwidth. Here an n-vertex graph H has *bandwidth* b if there exists an ordering $v_1, \dots, v_n$ of $V(H)$ such that all edges $v_iv_j \in E(H)$ satisfy $|i - j| \leq b$. Throughout the paper, by H being k-chromatic we mean $\chi(H) \leq k$. This conjecture was resolved by the bandwidth theorem of Böttcher, Schacht and Taraz [9]. Note that while this result is essentially best possible when considering the class of k-chromatic graphs as a whole (consider e.g. K_k-factors), the results in [1, 38] mentioned above show that there are many graphs H for which the actual threshold is significantly smaller (e.g. the C_5-factors mentioned above).

The notion of bandwidth is related to the concept of separability: An n-vertex graph H is said to be η*-separable* if there exists a set S of at most ηn vertices such that every component of $H \setminus S$ has size at most ηn. We call such a set an η*-separator* of H. In general, the notion of having small bandwidth is more restrictive than that of being separable. However, for graphs with bounded maximum degree, it turns out that these notions are actually equivalent (see [8]).

1.2. Previous results: (approximate) decompositions into large graphs.

We say that a collection $\mathcal{H} = \{H_1, \dots, H_s\}$ of graphs *packs* into G if there exist pairwise edge-disjoint copies of $H_1, \dots, H_s$ in G. In cases where $\mathcal{H}$ consists of copies of a single graph H we refer to this packing as an H*-packing* in G. If $\mathcal{H}$ packs into G and $e(\mathcal{H}) = e(G)$ (where $e(\mathcal{H}) = \sum_{H \in \mathcal{H}} e(H)$), then we say that G has a *decomposition* into $\mathcal{H}$. Once again, if $\mathcal{H}$ consists of copies of a single graph H, we refer to this as an H-decomposition of G. Informally, we refer to a packing which covers almost all edges of the host graph G as an approximate decomposition.

As in the previous section, most attention so far has focussed on (Hamilton) cycles, trees, factors, and graphs of bounded degree. Indeed, a classical construction of Walecki going back to the 19th century guarantees a decomposition of K_n into Hamilton cycles whenever n is odd. As mentioned earlier, this was extended to Hamilton decompositions of regular graphs G of high degree by Csaba, Kühn, Lo, Osthus and Treglown [12] (based on the existence of Hamilton decompositions in robustly expanding graphs proved in [37]). A different generalisation of Walecki's construction is given by the Alspach problem, which asks for a decomposition of K_n into cycles of given length. This was recently resolved by Bryant, Horsley and Petterson [10].

A further famous open problem in the area is the tree packing conjecture of Gyárfás and Lehel, which says that for any collection $\mathcal{T} = \{T_1, \dots, T_n\}$ of trees with $|V(T_i)| = i$, the complete graph K_n has a decomposition into $\mathcal{T}$. This was recently proved by Joos, Kim, Kühn and Osthus [27] for the case where n is large and each T_i has bounded degree. The crucial tool for this was the blow-up lemma for approximate decompositions of ε-regular graphs G by Kim, Kühn, Osthus and Tyomkyn [30]. In particular, this lemma implies that if $\mathcal{H}$ is a family of bounded degree n-vertex graphs with $e(\mathcal{H}) \leq (1 - o(1))\binom{n}{2}$, then K_n has an approximate decomposition into $\mathcal{H}$. This generalises earlier results of Böttcher, Hladkỳ, Piguet and Taraz [7] on tree packings, as well as results of Messuti, Rödl and Schacht [39] and Ferber, Lee and Mousset [17] on packing separable graphs. Very recently, Allen, Böttcher, Hladkỳ and Piguet [2] were able to show that one can in fact find an approximate decomposition of K_n into $\mathcal{H}$ provided that the graphs in $\mathcal{H}$ have bounded degeneracy and maximum degree $o(n/\log n)$. This implies an approximate version of the tree packing conjecture when the trees have maximum degree $o(n/\log n)$. The latter improves a bound of Ferber and Samotij [18] which follows from their work on packing (spanning) trees in random graphs.

An important type of decomposition of K_n is given by resolvable designs: a resolvable F-design consists of a decomposition into F-factors. Ray-Chaudhuri and Wilson [42] proved the existence of resolvable K_k-designs in K_n (subject to the necessary divisibility conditions being satisfied). This was generalised to arbitrary F-designs by Dukes and Ling [16].

1.3. **Main result: packing separable graphs of bounded degree.** Our main result provides a degree condition which ensures that G has an approximate decomposition into $\mathcal{H}$ for any collection $\mathcal{H}$ of k-chromatic η-separable graphs of bounded degree. As discussed below, our degree condition is best possible in general (unless one has additional information about the graphs in $\mathcal{H}$). By the remark at the end of Section 1.1 earlier, one can replace the condition of being η-separable by that of having bandwidth at most ηn in Theorem 1.2. Thus our result implies a version of the bandwidth theorem of [9] in the setting of approximate decompositions.

To state our result, we first introduce the approximate K_k-decomposition threshold δ_k^{reg} for regular graphs.

Definition 1.1 (Approximate K_k-decomposition threshold for regular graphs)**.** *For each $k \in \mathbb{N}\backslash\{1\}$, let δ_k^{reg} be the infimum over all $\delta \geq 0$ satisfying the following: for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and $r \geq \delta n$ every n-vertex r-regular graph G has a K_k-packing consisting of at least $(1-\varepsilon)e(G)/e(K_k)$ copies of K_k.*

Roughly speaking, we will pack k-chromatic graphs H into regular host graphs G of degree at least $\delta_k^{\mathrm{reg}} n$. Actually it turns out that it suffices to assume that H is 'almost' k-chromatic in the sense that H has a $(k+1)$-colouring where one colour is used only rarely. More precisely, we say that H is (k,η)-*chromatic* if there exists a proper colouring of the graph H' obtained from H by deleting all its isolated vertices with $k+1$ colours such that one of the colour classes has size at most $\eta|V(H')|$. A similar feature is also present in [9].

Theorem 1.2. *For all $\Delta, k \in \mathbb{N}\backslash\{1\}$, $0 < \nu < 1$ and $\max\{1/2, \delta_k^{reg}\} < \delta \leq 1$, there exist $\xi, \eta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following holds. Suppose that $\mathcal{H}$ is a collection of n-vertex (k,η)-chromatic η-separable graphs and G is an n-vertex graph such that*

(i) $(\delta-\xi)n \leq \delta(G) \leq \Delta(G) \leq (\delta+\xi)n$,
(ii) $\Delta(H) \leq \Delta$ *for all* $H \in \mathcal{H}$,
(iii) $e(\mathcal{H}) \leq (1-\nu)e(G)$.

Then $\mathcal{H}$ packs into G.

Note that our result holds for any minor-closed family $\mathcal{H}$ of k-chromatic bounded degree graphs by the separator theorem of Alon, Seymour and Thomas [3]. Moreover, note that since $\mathcal{H}$ may consist e.g. of Hamilton cycles, the condition that G is close to regular is clearly necessary. Also, the condition $\max\{1/2, \delta_k^{\mathrm{reg}}\} < \delta$ is necessary. To see this, if $\delta_k^{\mathrm{reg}} \leq 1/2$ (which holds if $k=2$), then we consider $K_{n/2-1,n/2+1}$ which does not even contain a single perfect matching, let alone an approximate decomposition into perfect matchings. If $\delta_k^{\mathrm{reg}} > 1/2$ (which holds if $k \geq 3$), then for any $\delta < \delta_k^{\mathrm{reg}}$, the definition of δ_k^{reg} ensures that there exist arbitrarily large regular graphs G of degree at least δn without an approximate decomposition into copies of K_k. As a disjoint union of a single copy of K_k with $n-k$ isolated vertices satisfies (ii), this shows that the condition of $\max\{1/2, \delta_k^{\mathrm{reg}}\} < \delta$ is sharp when considering the class of all k-chromatic separable graphs (though as in the case of embedding a single copy of some H into G, it may be possible to improve the degree bound for certain families $\mathcal{H}$).

To obtain explicit estimates for δ_k^{reg}, we also introduce the approximate K_k-decomposition threshold δ_k^{0+} for graphs of large minimum degree.

Definition 1.3 (Approximate K_k-decomposition threshold)**.** *For each $k \in \mathbb{N}\setminus\{1\}$, let δ_k^{0+} be the infimum over all $\delta \geq 0$ satisfying the following: for any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that any n-vertex graph G with $n \geq n_0$ and $\delta(G) \geq \delta n$ has a K_k-packing consisting of at least $(1-\varepsilon)e(G)/e(K_k)$ copies of K_k.*

It is easy to see that $\delta_2^{\mathrm{reg}} = \delta_2^{0+} = 0$ and $\delta_k^{\mathrm{reg}} \leq \delta_k^{0+}$. The value of δ_k^{0+} has been subject to much attention recently: one reason is that by results of [5, 19], for $k \geq 3$ the approximate decomposition threshold δ_k^{0+} is equal to the analogous threshold δ_k^{dec} which ensures a 'full' K_k-decomposition of any n-vertex graph G with $\delta(G) \geq (\delta_k^{\mathrm{dec}} + o(1))n$ which satisfies the necessary divisibility conditions. A beautiful conjecture (due to Nash-Williams in the triangle case and Gustavsson in the general case) would imply that $\delta_k^{\mathrm{dec}} = 1 - 1/(k+1)$ for $k \geq 3$. On the other hand for $k \geq 3$, it is easy to modify a well-known construction (see Proposition 3.7) to show that $\delta_k^{\mathrm{reg}} \geq 1 - 1/(k+1)$. Thus the conjecture would imply that $\delta_k^{\mathrm{reg}} = \delta_k^{0+} = \delta_k^{\mathrm{dec}} = 1 - 1/(k+1)$ for $k \geq 3$. A result of Dross [15] implies that $\delta_3^{0+} \leq 9/10$, and a very recent result of Montgomery [40] implies that $\delta_k^{0+} \leq 1 - 1/(100k)$ (see Lemma 3.10). With these bounds, the following corollary is immediate.

Corollary 1.4. *For all $\Delta, k \in \mathbb{N}\setminus\{1\}$ and $0 < \nu, \delta < 1$, there exist $\xi > 0$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ the following holds for every n-vertex graph G with*

$$(\delta - \xi)n \leq \delta(G) \leq \Delta(G) \leq (\delta + \xi)n.$$

(i) Let $\mathcal{T}$ be a collection of trees such that for all $T \in \mathcal{T}$ we have $|T| \leq n$ and $\Delta(T) \leq \Delta$. Further suppose $\delta > 1/2$ and $e(\mathcal{T}) \leq (1-\nu)e(G)$. Then $\mathcal{T}$ packs into G.
(ii) Let F be an n-vertex graph consisting of a union of vertex-disjoint cycles and let $\mathcal{F}$ be a collection of copies of F. Further suppose $\delta > 9/10$ and $e(\mathcal{F}) \leq (1-\nu)e(G)$. Then $\mathcal{F}$ packs into G.
(iii) Let $\mathcal{C}$ be a collection of cycles, each on at most n vertices. Further suppose $\delta > 9/10$ and $e(\mathcal{C}) \leq (1-\nu)e(G)$. Then $\mathcal{C}$ packs into G.
(iv) Let n be divisible by k and let $\mathcal{K}$ be a collection of n-vertex K_k-factors. Further suppose $\delta > 1 - 1/(100k)$ and $e(\mathcal{K}) \leq (1-\nu)e(G)$. Then $\mathcal{K}$ packs into G.

Note that (i) can be viewed as an approximate version of the tree packing conjecture in the setting of dense (almost) regular graphs. In a similar sense, (ii) relates to the Oberwolfach conjecture, (iii) relates to the Alspach problem and (iv) relates to the existence of resolvable designs in graphs.

Moreover, the feature that Theorem 1.2 allows us to efficiently pack (k,η)-chromatic graphs (rather than k-chromatic graphs) gives several additional consequences, for example: if the cycles of F in (ii) are all sufficiently long, then we can replace the condition '$\delta > 9/10$' by '$\delta > 1/2$'.

If we drop the assumption of being G close to regular, then one can still ask for the size of the largest packing of bounded degree separable graphs. For example, it was shown in [12] that every sufficiently large graph G with $\delta(G) \geq n/2$ contains at least $(n-2)/8$ edge-disjoint Hamilton cycles. The following result gives an approximate answer to the above question in the case when $\mathcal{H}$ consists of (almost) bipartite graphs.

Theorem 1.5. *For all $\Delta \in \mathbb{N}$, $1/2 < \delta \leq 1$ and $\nu > 0$, there exist $\eta > 0$ and $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following holds. Suppose that $\mathcal{H}$ is a collection of n-vertex $(2,\eta)$-chromatic η-separable graphs and G is an n-vertex graph such that*

(i) $\delta(G) \geq \delta n$,
(ii) $\Delta(H) \leq \Delta$ for all $H \in \mathcal{H}$,
(iii) $e(\mathcal{H}) \leq \frac{(\delta+\sqrt{2\delta-1}-\nu)n^2}{4}$.

Then $\mathcal{H}$ packs into G.

The result in general cannot be improved: Indeed, for $\delta > 1/2$ the number of edges of the densest regular spanning subgraph of G is close to $(\delta + \sqrt{2\delta-1})n^2/4$ (see [11]). So the bound in (iii) is asymptotically optimal e.g. if n is even and $\mathcal{H}$ consists of Hamilton cycles. We discuss the very minor modifications to the proof of Theorem 1.2 which give Theorem 1.5 at the end of Section 6.

We raise the following open questions:

- We conjecture that the error term $\nu e(G)$ in condition (iii) of Theorem 1.2 can be improved. Note that it cannot be completely removed unless one assumes some divisibility conditions on G. However, even additional divisibility conditions will not always ensure a 'full' decomposition under the current degree conditions: indeed, for C_4, the minimum degree threshold which guarantees a C_4-decomposition of a graph G is close to $2n/3$, and the extremal example is close to regular (see [5] for details, more generally, the decomposition threshold of an arbitrary bipartite graph is determined in [19]).
- It would be interesting to know whether the condition on separability can be omitted. Note however, that if we do not assume separability, then the degree condition may need to be strengthened.
- It would be interesting to know whether one can relax the maximum degree condition in assumption (ii) of Theorem 1.2, e.g. for the class of trees.
- Given the recent progress on the existence of decompositions and designs in the hypergraph setting and the corresponding minimum degree thresholds [29, 20, 21], it would be interesting to generalise (some of) the above results to hypergraphs.

Our main tool in the proof of Theorem 1.2 will be the recent blow-up lemma for approximate decompositions by Kim, Kühn, Osthus and Tyomkyn [30]: roughly speaking, given a set $\mathcal{H}$ of n-vertex bounded degree graphs and an n-vertex graph G with $e(\mathcal{H}) \leq (1-o(1))e(G)$ consisting of super-regular pairs, it guarantees a packing of $\mathcal{H}$ in G (such super-regular pairs arise from applications of Szemerédi's regularity lemma). Theorem 3.15 gives the precise statement of the special case that we shall apply (note that the original blow-up lemma of Komlós, Sárközy and Szemerédi [31] corresponds to the case where $\mathcal{H}$ consists of a single graph).

Subsequently, Theorem 1.2 has been used as a key tool in the resolution of the Oberwolfach problem in [22]. This was posed by Ringel in 1967, given an n-vertex graph H consisting of vertex-disjoint cycles, it asks for a decomposition of K_n into copies of H (if n is odd). In fact, the results in [22] go considerably beyond the setting of the Oberwolfach problem, and imply e.g. a positive resolution also to the Hamilton-Waterloo problem.

2. Outline of the argument

Consider a given collection $\mathcal{H}$ of k-chromatic η-separable graphs with bounded degree and a given almost-regular graph G as in Theorem 1.2. We wish to pack $\mathcal{H}$ into G. The approach will be to decompose G into a bounded number of highly structured subgraphs G_t and partition $\mathcal{H}$ into a bounded number of collections $\mathcal{H}_t$. We then aim to pack each $\mathcal{H}_t$ into G_t. As described below, for each $H \in \mathcal{H}_t$, most of the edges will be embedded via the blow-up lemma for approximate decompositions proved in [30].

As a preliminary step, we first apply Szemerédi's regularity lemma (Lemma 3.5) to G to obtain a reduced multigraph R which is almost regular. Here each edge e of R corresponds to a bipartite ε-regular subgraph of G and the density of these subgraphs does not depend on e. We can then apply a result of Pippenger and Spencer on the chromatic index of regular hypergraphs and the definition of δ_k^{reg} to find an approximate decomposition of the reduced multigraph R into almost K_k-factors. More precisely, we find a set of edge-disjoint copies of almost K_k-factors covering almost all edges of R, where an almost K_k-factor is a set of vertex-disjoint copies of K_k covering almost all vertices of R. This approximate decomposition translates into the existence of an approximate decomposition of G into '(almost-)K_k-factor blow-ups'. Here a K_k-factor blow-up consists of a bounded number of clusters $V_1, \dots, V_{kr}$ where each pair (V_i, V_j) with $\lfloor (i-1)/k \rfloor = \lfloor (j-1)/k \rfloor$ is ε-regular of density d, and crucially d does not depend on i, j. We wish to use the blow-up lemma for approximate decompositions (Theorem 3.15) to pack graphs into each K_k-factor blow-up. Ideally, we would like to split $\mathcal{H}$ into a bounded number of subcollections $\mathcal{H}_{t,s}$ and pack each $\mathcal{H}_{t,s}$ into a separate K_k-factor blow-up $G_{t,s}$, where the $G_{t,s} \subseteq G$ are all edge-disjoint.

There are several obstacles to this approach. The first obstacle is that (i) the K_k-factor blow-ups $G_{t,s}$ are not spanning. In particular, they do not contain the vertices in the exceptional set V_0 produced by the regularity lemma. On the other hand, if we aim to embed an n-vertex graph $H \in \mathcal{H}$ into G, we must embed some vertices of H into V_0. However, Theorem 3.15 does not produce an embedding into vertices outside the K_k-factor blow-up. The second obstacle is that (ii) the K_k-factor blow-ups are not connected, whereas H may certainly be (highly) connected. This is one significant difference to [9], where the existence of a structure similar to a blown-up power of a Hamilton path in R could be utilised for the embedding. A third issue is that (iii) any resolution of (i) and (ii) needs to result in a 'balanced' packing of the $H \in \mathcal{H}$, i.e. the condition $e(\mathcal{H}) \leq (1-\nu)e(G)$ means that for most $x \in V(G)$ almost all their incident edges need to be covered.

To overcome the first issue, we use the fact that H is η-separable to choose a small separating set S for H and consider the small components of $H - S$. To be able to embed (most of) H into the K_k-factor blow-up, we need to add further edges to each K_k-factor blow-up so that the resulting 'augmented K_k-factor blow-ups' have strong connectivity properties. For this, we partition $V(G)\backslash V_0$ into T disjoint 'reservoirs' $Res_1, \dots, Res_T$, where $1/T \ll 1$. We will later embed some vertices of H into V_0 using the edges between Res_t and V_0 (see Lemma 4.1). Here we have to embed a vertex of H onto $v \in V_0$ using only edges between v and Res_t because we do not have any control on the edges between v and a regularity cluster V_i. We explain the reason for choosing a partition into many reservoir sets (rather than choosing a single small reservoir) below.

We also decompose most of G into graphs $G_{t,s}$ so that each $G_{t,s}$ has vertex set $V(G)\backslash(Res_t \cup V_0)$ and is a K_k-factor blow-up. We then find sparse bipartite graphs $F_{t,s} \subseteq G$ connecting Res_t with $G_{t,s}$, bipartite graphs $F'_t \subseteq G$ connecting Res_t with V_0 as well as sparse graphs $G^*_t \subseteq G$ which provide connectivity within Res_t as well as between Res_t and $G_{t,s}$. The fact that $G_{t,s}$ and $G_{t,s'}$ share the same reservoir for $s \neq s'$ permits us to choose the reservoir Res_t to be significantly larger than V_0. Moreover, as $\bigcup Res_t$ covers all vertices in $V \backslash V_0$, if the graphs F'_t are appropriately chosen, then almost all edges incident to the vertices in V_0 are available to be used at some stage of the packing process. Our aim is to pack each $\mathcal{H}_{t,s}$ into the 'augmented' K_k-factor blow-up $G_{t,s} \cup F_{t,s} \cup F'_t \cup G^*_t$. To ensure that the resulting packings can be combined into a packing of all of the graphs in $\mathcal{H}$, we will use the fact that the graphs $G_t := \bigcup_s (G_{t,s} \cup F_{t,s}) \cup F'_t \cup G^*_t$ referred to in the first paragraph are edge-disjoint for different t.

We now discuss how to find this packing of $\mathcal{H}_{t,s}$. Consider some $H \in \mathcal{H}_{t,s}$. We first use the fact that H is separable to find a partition of H which reflects the structure of (the augmentation of) $G_{t,s}$ (see Section 4). Then we construct an appropriate embedding ϕ_* of parts of each graph $H \in \mathcal{H}_{t,s}$ into $Res_t \cup V_0$ which covers all vertices in $Res_t \cup V_0$ (this makes crucial use of the fact that Res_t is much larger than V_0). Later we aim to use the blow-up lemma for approximate decompositions (Theorem 3.15) to find an embedding ϕ of the remaining vertices of H into $V(G)\backslash(Res_t \cup V_0)$. When we apply Theorem 3.15, we use its additional features: in particular, the ability to prescribe appropriate 'target sets' for some of the vertices of H, to guarantee the consistency between the two embeddings ϕ_* and ϕ.

An important advantage of the reservoir partition which helps us to overcome obstacle (iii) is the following: the blow-up lemma for approximate decompositions can achieve a near optimal packing, i.e. it uses up almost all available edges. This is far from being the case for the part of the embeddings that use $F_{t,s}$, F'_t and G^*_t to embed vertices into $Res_t \cup V_0$, where the edge usage might be comparatively 'imbalanced' and 'inefficient'. (In fact, we will try to avoid using these edges as much as possible in order to preserve the connectivity properties of these graphs. We will use probabilistic allocations to avoid over-using any parts of $F_{t,s}$, F'_t and G^*_t.) However, since every vertex in $V(G_0)\backslash V_0$ is a reservoir vertex for only a small proportion of the embeddings, the resulting effect of these imbalances on the overall leftover degree of the vertices in $V(G_0)\backslash V_0$ is negligible. For V_0, we will be able to assign only low degree vertices of each H to ensure that there will always be edges of F'_t available to embed their incident edges (so the overall leftover degree of the vertices in V_0 may be large).

The above discussion motivates why we use many reservoir sets which cover all vertices in $V(G)\backslash V_0$, rather than using only one vertex set Res_1 for all $H \in \mathcal{H}$. Indeed, if some vertices of G only perform the role of reservoir vertices, this might result in an imbalance of the usage of edges incident to these vertices: some vertices in the reservoir might lose incident edges much faster or slower than the vertices in the regularity clusters. Apart from the fact that a fast loss of the edges incident to one vertex can prevent us from embedding any further spanning graphs into G, a large loss of the edges incident to the reservoir is also problematic in its own right. Indeed, since we are forced to use the edges incident to the reservoir in order to be able to embed some vertices onto vertices in V_0, this would prevent us from packing any further graphs.

Another issue is that the regularity lemma only gives us ε-regular K_k-factor blow-ups while we need super-regular K_k-factor blow-ups in order to use Theorem 3.15. To overcome this issue, we will make appropriate adjustments to each ε-regular K_k-factor blow-up. This means that the exceptional set V_0 will actually be different for each pair t, s of indices. We can however use probabilistic arguments to ensure that this does not significantly affect the overall 'balance' of the packing. In particular, for simplicity, in the above proof sketch we have ignored this issue.

The paper is organised as follows. We collect some basic tools in Section 3, and we prove a lemma which finds a suitable partition of each graph $H \in \mathcal{H}$ in Section 4 (Lemma 4.1). We prove our main lemma (Lemma 5.1) in Section 5. This lemma guarantees that we can find a suitable packing of an appropriate collection $\mathcal{H}_{t,s}$ of k-chromatic η-separable graphs with bounded degree into a graph consisting of a super-regular K_k-factor blow-up $G_{t,s}$ and suitable connection graphs $F_{t,s}$, F'_t and G^*_t. In Section 6, we will partition G and $\mathcal{H}$ as described above. Then we will repeatedly apply Lemma 5.1 to construct a packing of $\mathcal{H}$ into G.

3. Preliminaries

3.1. **Notation.** We write $[t] := \{1, \dots, t\}$. We often treat large numbers as integers whenever this does not affect the argument. The constants in the hierarchies used to state our results are chosen from right to left. That is, if we claim that a result holds for $0 < 1/n \ll a \ll b \leq 1$, we mean there exist non-decreasing functions $f : (0, 1] \to (0, 1]$ and $g : (0, 1] \to (0, 1]$ such that the result holds for all $0 \leq a, b \leq 1$ and all $n \in \mathbb{N}$ with $a \leq f(b)$ and $1/n \leq g(a)$. We will not calculate these functions explicitly.

We use the word *graphs* to refer to simple undirected finite graphs, and refer to *multi-graphs* as graphs with potentially parallel edges, but without loops. *Multi-hypergraphs* refer to (not necessarily uniform) hypergraphs with potentially parallel edges. A *k-graph* is a k-uniform hypergraph. A *multi-k-graph* is a k-uniform hypergraph with potentially parallel edges. For a multi-hypergraph H and a non-empty set $Q \subseteq V(H)$, we define $\mathrm{mult}_H(Q)$ to be the number of parallel edges of H consisting of exactly the vertices in Q. We say that a multi-hypergraph has *edge-multiplicity* at most t if $\mathrm{mult}_H(Q) \leq t$ for all non-empty $Q \subseteq V(H)$. A *matching* in a multi-hypergraph H is a collection of pairwise disjoint edges of H. The *rank* of a multi-hypergraph H is the size of a largest edge.

We write $H \simeq G$ if two graphs H and G are isomorphic. For a collection $\mathcal{H}$ of graphs, we let $v(\mathcal{H}) := \sum_{H\in\mathcal{H}} |V(H)|$. We say a partition $V_1, \dots, V_k$ of a set V is an *equipartition* if $||V_i| - |V_j|| \leq 1$ for all $i, j \in [k]$. For a multi-hypergraph H and $A, B \subseteq V(H)$, we let $E_H(A, B)$ denote the set of edges in H intersecting both A and B. We define $e_H(A, B) := |E_H(A, B)|$. For $v \in V(H)$ and $A \subseteq V(H)$, we let $d_{H,A}(v) := |\{e \in E(H) : v \in e, e\backslash\{v\} \subseteq A\}|$. Let $d_H(v) := d_{H,V(H)}(v)$. For $u, v \in V(H)$, we define $c_H(u, v) := |\{e \in E(H) : \{u, v\} \subseteq e\}|$. Let $\Delta(H) = \max\{d_H(v) : v \in V(H)\}$ and $\delta(H) := \min\{d_H(v) : v \in V(H)\}$.

For a graph G and sets $X, A \subseteq V(G)$, we define

$$N_{G,A}(X) := \{w \in A : uw \in E(G) \text{ for all } u \in X\} \text{ and } N_G(X) := N_{G,V(G)}(X).$$

Thus $N_G(X)$ is the common neighbourhood of X in G and $N_{G,A}(\emptyset) = A$. For a set $X \subseteq V(G)$, we define $N^d_G(X) \subseteq V(G)$ to be the set of all vertices of distance at most d from a vertex in X. In particular, $N^d_G(X) = \emptyset$ for $d < 0$. Note that $N_G(X)$ and $N^1_G(X)$ are different in general as e.g. vertices with a single edge to X are included in the latter. Moreover, note that $N_G(X) \subseteq N^1_G(X)$. We say a set $I \subseteq V(G)$ in a graph G is *k-independent* if for any two distinct vertices $u, v \in I$, the distance between u and v in G is at least k (thus a 2-independent set I is an independent set). If A, $B \subseteq V(G)$ are disjoint, we write $G[A, B]$ for the bipartite subgraph of G with vertex classes A, B and edge set $E_G(A, B)$.

For two functions $\phi : A \to B$ and $\phi' : A' \to B'$ with $A \cap A' = \emptyset$, we let $\phi \cup \phi'$ be the function from $A \cup A'$ to $B \cup B'$ such that for each $x \in A \cup A'$,

$$(\phi \cup \phi')(x) := \begin{cases} \phi(x) & \text{if } x \in A, \\ \phi'(x) & \text{if } x \in A'. \end{cases}$$

For graphs H and R with $V(R) \subseteq [r]$ and an ordered partition $(X_1, \dots, X_r)$ of $V(H)$, we say that H *admits the vertex partition* $(R, X_1, \dots, X_r)$, if $H[X_i]$ is empty for all $i \in [r]$, and for any $i, j \in [r]$ with $i \neq j$ we have that $e_H(X_i, X_j) > 0$ implies $ij \in E(R)$. We say that H is *internally q-regular with respect to* $(R, X_1, \dots, X_r)$ if H admits $(R, X_1, \dots, X_r)$ and $H[X_i, X_j]$ is q-regular for each $ij \in E(R)$.

We will often use the following Chernoff bound (see e.g. Theorem A.1.16 in [4]).

Lemma 3.1. [4] *Suppose $X_1, \dots, X_n$ are independent random variables such that $0 \leq X_i \leq b$ for all $i \in [n]$. Let $X := X_1 + \dots + X_n$. Then for all $t > 0$, $\mathbb{P}[|X - \mathbb{E}[X]| \geq t] \leq 2e^{-t^2/(2b^2n)}$.*

3.2. **Tools involving ε-regularity.** In this subsection, we introduce the definitions of (ε, d)-regularity and (ε, d)-super-regularity. We then state a suitable form of the regularity lemma for our purpose. We will also state an embedding lemma (Lemma 3.6) which we will use later to prove our main lemma (Lemma 5.1).

We say that a bipartite graph G with vertex partition (A, B) is *(ε, d)-regular* if for all sets $A' \subseteq A$, $B' \subseteq B$ with $|A'| \geq \varepsilon|A|$, $|B'| \geq \varepsilon|B|$, we have $|\frac{e_G(A',B')}{|A'||B'|} - d| < \varepsilon$. Moreover, we say that G is *ε-regular* if it is (ε, d)-regular for some d. If G is (ε, d)-regular and $d_G(a) = (d \pm \varepsilon)|B|$ for $a \in A$ and $d_G(b) = (d \pm \varepsilon)|A|$ for $b \in B$, then we say G is *(ε, d)-super-regular*. We say that G is *$(\varepsilon, d)^+$-(super)-regular* if it is (ε, d')-(super)-regular for some $d' \geq d$.

For a graph R on vertex set $[r]$, and disjoint vertex subsets $V_1, \dots, V_r$ of $V(G)$, we say that G is *$(\varepsilon, d)^+$-(super)-regular with respect to the vertex partition* $(R, V_1, \dots, V_r)$ if $G[V_i, V_j]$ is $(\varepsilon, d)^+$-(super)-regular for all $ij \in E(R)$. Being *(ε, d)-(super)-regular with respect to the vertex partition* $(R, V_1, \dots, V_r)$ is defined analogously. The following observations follow directly from the definitions.

Proposition 3.2. *Let $0 < \varepsilon \leq \delta \leq d \leq 1$. Suppose G is an (ε, d)-regular bipartite graph with vertex partition (A, B) and let $A' \subseteq A$, $B' \subseteq B$ with $|A'|/|A|$, $|B'|/|B| \geq \delta$. Then $G[A', B']$ is $(\varepsilon/\delta, d)$-regular.*

Proposition 3.3. *Let $0 < \varepsilon \leq \delta \leq d \leq 1$. Suppose G is an (ε, d)-regular bipartite graph with vertex partition (A, B). If G' is a subgraph of G with $V(G') = V(G)$ and $e(G') \geq (1 - \delta)e(G)$, then G' is $(\varepsilon + \delta^{1/3}, d)$-regular.*

Proposition 3.4. *Let $0 < \varepsilon \ll d \leq 1$. Suppose G is an (ε, d)-regular bipartite graph with vertex partition (A, B). Let*

$$A' := \{a \in A : d_G(a) \neq (d \pm \varepsilon)|B|\} \text{ and } B' := \{b \in B : d_G(b) \neq (d \pm \varepsilon)|B|\}.$$

Then $|A'| \leq 2\varepsilon|A|$ and $|B'| \leq 2\varepsilon|B|$.

The next lemma is a 'degree version' of Szemerédi's regularity lemma (see e.g. [36] on how to derive it from the original version).

Lemma 3.5 (Szemerédi's regularity lemma)**.** *Suppose $M, M', n \in \mathbb{N}$ with $0 < 1/n \ll 1/M \ll \varepsilon, 1/M' < 1$ and $d > 0$. Then for any n-vertex graph G, there exist a partition of $V(G)$ into $V_0, V_1, \dots, V_r$ and a spanning subgraph $G' \subseteq G$ satisfying the following.*

(i) $M' \leq r \leq M$,
(ii) $|V_0| \leq \varepsilon n$,
(iii) $|V_i| = |V_j|$ *for all* $i, j \in [r]$,
(iv) $d_{G'}(v) > d_G(v) - (d + \varepsilon)n$ *for all* $v \in V(G)$,
(v) $e(G'[V_i]) = 0$ *for all* $i \in [r]$,

(vi) *for all i, j with $1 \le i < j \le r$, the graph $G'[V_i, V_j]$ is either empty or $(\varepsilon, d_{i,j})$-regular for some $d_{i,j} \in [d, 1]$.*

The next lemma allows us to embed a small graph H into a graph G which is $(\varepsilon, d)^+$-regular with respect to a suitable vertex partition $(R, V_1, \dots, V_r)$. In our proof of Lemma 5.1 later on, properties $(\mathrm{B1})_{3.6}$ and $(\mathrm{B2})_{3.6}$ will help us to prescribe appropriate 'target sets' for some of the vertices when we apply the blow-up lemma for approximate decompositions (Theorem 3.15). There, H will be part of a larger graph that is embedded in several stages. $(\mathrm{B1})_{3.6}$ ensures that the embedding of H is compatible with constraints arising from earlier stages and $(\mathrm{B2})_{3.6}$ will ensure the existence of sufficiently large target sets when embedding vertices x in later stages (each edge of $\mathcal{M}$ corresponds to the neighbourhood of such a vertex x).

Lemma 3.6. *Suppose $n, \Delta \in \mathbb{N}$ with $0 < 1/n \ll \varepsilon \ll \alpha, \beta, d, 1/\Delta \le 1$. Suppose that G, H are graphs and $\mathcal{M}$ is a multi-hypergraph on $V(H)$ with edge-multiplicity at most Δ. Suppose $V_1, \dots, V_r$ are pairwise disjoint subsets of $V(G)$ with $\beta n \le |V_i| \le n$ for all $i \in [r]$, and $X_1, \dots, X_r$ is a partition of $V(H)$ with $|X_i| \le \varepsilon n$ for all $i \in [r]$. Let $f : E(\mathcal{M}) \to [r]$ be a function, and for all $i \in [r]$ and $x \in X_i$, let $A_x \subseteq V_i$. Let R be a graph on $[r]$. Suppose that the following hold.*

$(\mathrm{A1})_{3.6}$ *G is $(\varepsilon, d)^+$-regular with respect to $(R, V_1, \dots, V_r)$,*
$(\mathrm{A2})_{3.6}$ *H admits the vertex partition $(R, X_1, \dots, X_r)$,*
$(\mathrm{A3})_{3.6}$ *$\Delta(H) \le \Delta$, $\Delta(\mathcal{M}) \le \Delta$ and the rank of $\mathcal{M}$ is at most Δ,*
$(\mathrm{A4})_{3.6}$ *for all $i, j \in [r]$, if $f(e) = i$ and $e \cap X_j \neq \emptyset$, then $ij \in E(R)$,*
$(\mathrm{A5})_{3.6}$ *for all $i \in [r]$ and $x \in X_i$, we have $|A_x| \ge \alpha |V_i|$.*

Then there exists an embedding ϕ of H into G such that

$(\mathrm{B1})_{3.6}$ *for each $x \in V(H)$, we have $\phi(x) \in A_x$,*
$(\mathrm{B2})_{3.6}$ *for each $e \in \mathcal{M}$, we have $|N_G(\phi(e)) \cap V_{f(e)}| \ge (d/2)^\Delta |V_{f(e)}|$.*

Note that $(\mathrm{A4})_{3.6}$ implies for all $e \in E(\mathcal{M})$ that $e \cap X_{f(e)} = \emptyset$.

Proof. For each $x \in V(H)$, let $e_x := N_H(x)$ and $\mathcal{M}'$ be a multi-hypergraph on vertex set $V(H)$ with $E(\mathcal{M}') = \{e_x : x \in V(H)\}$. Since a vertex $x \in V(H)$ belongs to e_y only when $y \in N_H(x)$, we have $d_{\mathcal{M}'}(x) = d_H(x)$. So $\mathcal{M}'$ is a multi-hypergraph with rank at most Δ and $\Delta(\mathcal{M}') \le \Delta$. Let $\mathcal{M}^* := \mathcal{M} \cup \mathcal{M}'$ and for each $e \in E(\mathcal{M}^*)$, define

$$B_e := \begin{cases} V_{f(e)} & \text{if } e \in E(\mathcal{M}), \\ A_x & \text{if } e = e_x \in E(\mathcal{M}') \text{ for } x \in V(H). \end{cases}$$

Note that by $(\mathrm{A3})_{3.6}$, we have

$$\mathcal{M}^* \text{ has rank at most } \Delta, \text{ and } \Delta(\mathcal{M}^*) \le \Delta(\mathcal{M}) + \Delta(\mathcal{M}') \le 2\Delta. \tag{3.1}$$

Let $V(H) := \{x_1, \dots, x_m\}$, and for each $i \in [m]$, we let $Z_i := \{x_1, \dots, x_i\}$. We will iteratively extend partial embeddings $\phi_0, \dots, \phi_m$ of H into G in such a way that the following hold for all $i \le m$.

$(\Phi1)^i_{3.6}$ ϕ_i embeds $H[Z_i]$ into G,
$(\Phi2)^i_{3.6}$ $\phi_i(x_k) \in A_{x_k}$, for all $k \in [i]$,
$(\Phi3)^i_{3.6}$ for all $e \in \mathcal{M}^*$, we have $|N_G(\phi_i(e \cap Z_i)) \cap B_e| \ge (d/2)^{|e \cap Z_i|} |B_e|$.

Note that $(\Phi1)^0_{3.6}$–$(\Phi3)^0_{3.6}$ hold for an empty embedding $\phi_0 : \emptyset \to \emptyset$. Assume that for some $i \in [m]$, we have already defined an embedding ϕ_{i-1} satisfying $(\Phi1)^{i-1}_{3.6}$–$(\Phi3)^{i-1}_{3.6}$. We will construct ϕ_i by choosing an appropriate image for x_i. Let $s \in [r]$ be such that $x_i \in X_s$, and let $S := N_G(\phi_{i-1}(Z_i \cap e_{x_i})) \cap B_{e_{x_i}}$. Thus $S \subseteq V_s$. Since $Z_{i-1} \cap e_{x_i} = Z_i \cap e_{x_i}$, we have that $(\Phi3)^{i-1}_{3.6}$ implies

$$|S| \ge (d/2)^{|Z_i \cap e_{x_i}|} \alpha \beta n > (d/2)^\Delta \alpha \beta n > \varepsilon^{1/3} n. \tag{3.2}$$

For each $e \in E(\mathcal{M}^*)$ containing x_i, we consider

$$S_e := N_G(\phi_{i-1}(Z_{i-1} \cap e)) \cap B_e.$$

By $(\Phi3)^{i-1}_{3.6}$, we have

$$|S_e| \ge (d/2)^\Delta \alpha \beta n > \varepsilon^{1/3} n. \tag{3.3}$$

If $e = N_H(x)$ for some $x \in X_{s'}$ with $s' \in [r]$, then we have $S_e \subseteq B_e \subseteq V_{s'}$, and $(A2)_{3.6}$ implies that $ss' \in E(R)$. Moreover, note that if $e \in \mathcal{M}$ with $f(e) = s'$ for some $s' \in [r]$, then $S_e \subseteq B_e = V_{s'}$, and $(A4)_{3.6}$ implies that $ss' \in E(R)$. Thus in any case, $(A1)_{3.6}$ implies that $G[V_s, V_{s'}]$ is (ε, d')-regular for some $d' \geq d$. Hence, Proposition 3.2 with (3.2) and (3.3) implies that $G[S, S_e]$ is $(\varepsilon^{1/2}, d')$-regular. Let

$$S'_e := \{v \in S : d_{G,S_e}(v) < (d/2)|S_e|\}.$$

By Proposition 3.4, we have $|S'_e| \leq 2\varepsilon^{1/2}n$. Thus

$$|S \setminus \bigcup_{e\in E(\mathcal{M}^*):x_i\in e} S'_e| \overset{(3.1)}{\geq} |S| - 2\Delta \cdot 2\varepsilon^{1/2}n \overset{(3.2)}{\geq} 1. \tag{3.4}$$

We choose $v \in S \setminus \bigcup_{e\in E(\mathcal{M}^*):x_i\in e} S'_e$, and we extend ϕ_{i-1} into ϕ_i by letting $\phi_i(x_i) := v$. Since

$$\phi_i(x_i) \in S = N_G(\phi_{i-1}(Z_i \cap e_{x_i})) \cap B_{e_{x_i}} = N_G(\phi_i(Z_i \cap N_H(x_i))) \cap A_{x_i},$$

$(\Phi1)^i_{3.6}$ and $(\Phi2)^i_{3.6}$ hold. Also, for each $e \in E(\mathcal{M}^*)$, if $x_i \notin e$, then as we have $Z_i \cap e = Z_{i-1} \cap e$,

$$|N_G(\phi_i(Z_i \cap e)) \cap B_e| = |N_G(\phi_{i-1}(Z_{i-1} \cap e)) \cap B_e| \overset{(\Phi3)^{i-1}_{3.6}}{\geq} (d/2)^{|Z_i\cap e|}|B_e|.$$

If $x_i \in e$, then since $\phi_i(x_i) \notin S'_e$ and $|Z_i \cap e| = |Z_{i-1} \cap e| + 1$, we have

$$|N_G(\phi_i(Z_i \cap e)) \cap B_e| \quad \geq \quad |N_G(\phi_i(x_i)) \cap S_e| \geq (d/2)|S_e| \overset{(\Phi3)^{i-1}_{3.6}}{\geq} (d/2)^{|Z_i\cap e|}|B_e|. \tag{3.5}$$

Thus $(\Phi3)^i_{3.6}$ holds. By repeating this until we have embedded all vertices of H, we obtain an embedding ϕ_m satisfying $(\Phi1)^m_{3.6}$–$(\Phi3)^m_{3.6}$. Let $\phi := \phi_m$. Then $(\Phi2)^m_{3.6}$ implies that $(B1)_{3.6}$ holds, and $(\Phi3)^m_{3.6}$ together with $(A3)_{3.6}$ and the definition of B_e implies that $(B2)_{3.6}$ holds. □

3.3. **Decomposition tools.** In this subsection, we first give bounds on $\delta_k^{\rm reg}$. The following proposition provides a lower bound for $\delta_k^{\rm reg}$. The proof is only a slight extension of the extremal construction given by Proposition 1.5 in [5], and thus we omit it here.

Proposition 3.7. *For all $k \in \mathbb{N}\setminus\{1, 2\}$ we have $\delta_k^{reg} \geq 1 - 1/(k + 1)$.*

It will be convenient to use that for $k \geq 2$ this lower bound implies

$$\max\{1/2, \delta_k^{\rm reg}\} \geq 1 - 1/k. \tag{3.6}$$

Given two graphs F and G, let $\binom{G}{F}$ denote the set of all copies of F in G. A function ψ from $\binom{G}{F}$ to $[0, 1]$ is a *fractional F-packing* of G if $\sum_{F'\in\binom{G}{F}:e\in F'} \psi(F') \leq 1$ for each $e \in E(G)$ (if we have equality for each $e \in E(G)$ then this is referred to as a *fractional F-decomposition*). Let $\nu^*_F(G)$ be the maximum value of $\sum_{F'\in\binom{G}{F}} \psi(F')$ over all fractional F-packings ψ of G. Thus $\nu^*_F(G) \leq e(G)/e(F)$ and $\nu^*_F(G) = e(G)/e(F)$ if and only if G has a fractional F-decomposition. The following very recent result of Montgomery gives a degree condition which ensures a fractional K_k-decomposition in a graph.

Theorem 3.8. [40] *Suppose $k, n \in \mathbb{N}$ and $0 < 1/n \ll 1/k < 1$. Then any n-vertex graph G with $\delta(G) \geq (1 - 1/(100k))n$ satisfies $\nu^*_{K_k}(G) = e(G)/e(K_k)$.*

The next result due to Haxell and Rödl implies that a fractional K_k-decomposition gives rise to the existence of an approximate K_k-decomposition.

Theorem 3.9. [25] *Suppose $n \in \mathbb{N}$ with $0 < 1/n \ll \varepsilon < 1$. Then any n-vertex graph G has an F-packing consisting of at least $\nu^*_F(G) - \varepsilon n^2$ copies of F.*

Lemma 3.10. *For $k \in \mathbb{N}\setminus\{1, 2\}$, we have $\delta_k^{reg} \leq \delta_k^{0+} \leq 1 - 1/(100k)$. Moreover, $\delta_2^{reg} = \delta_2^{0+} = 0$ and $\delta_3^{reg} \leq \delta_3^{0+} \leq 9/10$.*

Proof. It is easy to see that Theorem 3.8 and Theorem 3.9 together imply that $\delta_k^{0+} \leq 1 - 1/(100k)$. Moreover, Theorem 3.9 together with a result of Dross [5] implies that $\delta_3^{0+} \leq 9/10$. As any graph can be decomposed into copies of K_2, we have $\delta_2^{0+} = 0$. □

In the remainder of this subsection, we prove Lemma 3.13. In the proof of Theorem 1.2, we will apply it to obtain an approximate decomposition of the reduced multi-graph R into almost K_k-factors (see Section 6). We will use the following consequence of Tutte's r-factor theorem.

Theorem 3.11. [11] *Suppose $n \in \mathbb{N}$ and $0 < 1/n \ll \gamma \ll 1$. If G is an n-vertex graph with $\delta(G) \geq (1/2+\gamma)n$ and $\Delta(G) \leq \delta(G) + \gamma^2 n$, then G contains a spanning r-regular subgraph for every even r with $r \leq \delta(G) - \gamma n$.*

The following powerful result of Pippenger and Spencer [41] (based on the Rödl nibble) shows that every almost regular multi-k-graph with small maximum codegree has small chromatic index.

Theorem 3.12. [41] *Suppose $n, k \in \mathbb{N}$ and $0 < 1/n \ll \mu \ll \varepsilon, 1/k < 1$. Suppose H is an n-vertex multi-k-graph satisfying $\delta(H) \geq (1-\mu)\Delta(H)$, and $c_H(u,v) \leq \mu\Delta(H)$ for all $u \neq v \in V(H)$. Then we can partition $E(H)$ into $(1+\varepsilon)\Delta(H)$ matchings.*

We can now combine these tools to approximately decompose an almost regular multi-graph G of sufficient degree into 'almost' K_k-factors. All vertices of G will be used in almost all these factors except the vertices in a 'bad' set V' which are not used in any factor. Moreover, the factors come in T groups of equal size such that parallel edges of G belong to different groups. As explained in Section 2, we will apply this to the reduced multi-graph obtained from Szemerédi's regularity lemma.

Lemma 3.13. *Suppose $n, k, q, T \in \mathbb{N}$ with $0 < 1/n \ll \varepsilon, \sigma, 1/T, 1/k, 1/q, \nu \leq 1/2$ and $0 < 1/n \ll \xi \ll \nu < \sigma/2 < 1$ and $\delta = \max\{1/2, \delta_k^{reg}\} + \sigma$ and q divides T. Let G be an n-vertex multi-graph with edge-multiplicity at most q, such that for all $v \in V(G)$ we have*

$$d_G(v) = (\delta \pm \xi)qn.$$

Then there exists a subset $V' \subseteq V(G)$ with $|V'| \leq \varepsilon n$ and k dividing $|V(G)\backslash V'|$, and there exist pairwise edge-disjoint subgraphs $F_{1,1}, \dots, F_{1,\kappa}, F_{2,1}, \dots, F_{T,\kappa}$ with $\kappa = (\delta - \nu \pm \varepsilon)\frac{qn}{T(k-1)}$ satisfying the following.

(B1)$_{3.13}$ *For each $(t', i) \in [T] \times [\kappa]$, we have that $V(F_{t',i}) \subseteq V(G)\backslash V'$ and $F_{t',i}$ is a vertex-disjoint union of at least $(1-\varepsilon)n/k$ copies of K_k,*
(B2)$_{3.13}$ *for each $v \in V(G) \setminus V'$, we have $|\{(t', i) \in [T] \times [\kappa] : v \in V(F_{t',i})\}| \geq T\kappa - \varepsilon n$,*
(B3)$_{3.13}$ *for all $t' \in [T]$ and $u, v \in V(G)$, we have $|\{i \in [\kappa] : u \in N_{F_{t',i}}(v)\}| \leq 1$.*

Proof. It suffices to prove the lemma for the case when $T = q$. The general case then follows by relabelling. (We can split each group obtained from the $T = q$ case into T/q equal groups arbitrarily.) We choose a new constant μ such that

$$1/n \ll \mu \ll \varepsilon, \xi, \sigma, 1/k, 1/q.$$

For an edge colouring $\phi : E(G) \to [q]$ and $c \in [q]$, we let $G^c \subseteq G$ be the subgraph with edge set $\{e \in E(G) : \phi(e) = c\}$. We wish to show that there exists an edge-colouring $\phi : E(G) \to [q]$ satisfying the following for all $v \in V(G)$ and $c \in [q]$:

(Φ1)$_{3.13}$ $d_{G^c}(v) = (\delta \pm 2\xi)n$,
(Φ2)$_{3.13}$ G^c is a simple graph.

Recall that $e_G(u,v)$ denotes the number of edges of G between u and v. For each $\{u, v\} \in \binom{V(G)}{2}$, we choose a set $A_{\{u,v\}}$ uniformly at random from $\binom{[q]}{e_G(u,v)}$. For each $e \in E(G)$, we let $\phi(e) \in [q]$ be such that ϕ is bijective between $E_G(u,v)$ and $A_{\{u,v\}}$. This ensures that (Φ2)$_{3.13}$ holds. It is easy to see that (Φ1)$_{3.13}$ also holds with high probability by using Lemma 3.1.

Since $\delta \geq 1/2 + \sigma$ and $\xi \ll \nu, \sigma$, Theorem 3.11 implies that, for each $c \in [q]$, there exists a $(\delta - \nu)n$-regular spanning subgraph G^c_* of G^c. (By adjusting ν slightly we may assume that $(\delta - \nu)n$ is an even integer.) Since $\delta - \nu > \delta_k^{\rm reg} + \sigma/2$ and $1/n \ll \mu$, the graph G^c_* has a K_k-packing $\mathcal{Q}^c := \{Q^c_1, \dots, Q^c_t\}$ of size

$$t := \frac{(\delta - \nu - \mu)n^2}{k(k-1)}. \tag{3.7}$$

For each $c \in [q]$, let $\mathcal{H}^c$ be the k-graph with $V(\mathcal{H}^c) = V(G^c_*)$ and $E(\mathcal{H}^c) := \{V(Q^c_i) : i \in [t]\}$. By construction of $\mathcal{H}^c$, we have

$$\Delta(\mathcal{H}^c) \leq \frac{\Delta(G^c_*)}{k-1} \leq \frac{(\delta-\nu)n}{k-1}. \tag{3.8}$$

As $\mathcal{Q}^c$ is a K_k-packing in G^c_*, any pair $\{u,v\} \in \binom{V(G)}{2}$ belongs to at most one edge in $\mathcal{H}^c$. Thus for $\{u,v\} \in \binom{V(G)}{2}$,

$$c_{\mathcal{H}^c}(u,v) \leq 1. \tag{3.9}$$

Let

$$V'' := \bigcup_{c\in[q]} \Big\{ v \in V(G) : |\{i \in [t] : v \in V(Q^c_i)\}| < \frac{1}{k-1}(\delta - \nu - \mu^{1/3})n \Big\},$$

and let V' be a set consisting of the union of V'' as well as at most $k-1$ vertices arbitrarily chosen from $V(G)\setminus V''$ such that k divides $|V(G)\setminus V'|$. Note that for each $c \in [q]$, we have

$$e(G^c_*) - e(\mathcal{Q}^c) \leq \frac{1}{2}(\delta-\nu)n^2 - \binom{k}{2}t \overset{(3.7)}{\leq} \mu n^2.$$

On the other hand, since G^c_* is a $(\delta-\nu)n$-regular graph, we have

$$\begin{aligned} |V'| &\leq k+1+\sum_{c\in[q]} \frac{1}{\mu^{1/3}n} \sum_{v \in V(G)} \big(d_{G^c_*}(v) - (k-1)d_{\mathcal{H}^c}(v)\big) \\ &= k+1+\sum_{c\in[q]} \frac{2(e(G^c_*)-e(\mathcal{Q}^c))}{\mu^{1/3}n} \leq \frac{3q\mu n^2}{\mu^{1/3}n} \leq \mu^{1/2}n. \end{aligned} \tag{3.10}$$

Let $\tilde{\mathcal{H}}^c$ be the k-graph with $V(\tilde{\mathcal{H}}^c) := V(G^c_*) \setminus V'$ and $E(\tilde{\mathcal{H}}^c) := \{e \in E(\mathcal{H}^c) : e \cap V' = \emptyset\}$. Note that for any $v \in V(\tilde{\mathcal{H}}^c) = V(\mathcal{H}^c) \setminus V'$,

$$d_{\tilde{\mathcal{H}}^c}(v) = d_{\mathcal{H}^c}(v) \pm \sum_{u \in V'} c_{\mathcal{H}^c}(u,v) \overset{(3.9)}{=} d_{\mathcal{H}^c}(v) \pm |V'| \overset{(3.10),(3.8)}{=} \frac{(\delta-\nu\pm 2\mu^{1/3})n}{k-1}. \tag{3.11}$$

Note that we obtain the final equality from the definition of V' and the assumption that $v \notin V'$. Thus for each $c \in [q]$, we have $\delta(\tilde{\mathcal{H}}^c) \geq (1-\mu^{1/4})\Delta(\tilde{\mathcal{H}}^c)$. Together with (3.9) and the fact that $1/n \ll \mu \ll \varepsilon, 1/k, 1/q$, this ensures that we can apply Theorem 3.12 to see that for each $c \in [q]$, $E(\tilde{\mathcal{H}}^c)$ can be partitioned into $\kappa' := \frac{(\delta-\nu+\varepsilon^3/q)n}{k-1}$ matchings $M^c_1, \ldots, M^c_{\kappa'}$. Let

$$\mathcal{M}^c := \{M^c_i : i \in [\kappa']\} \text{ and } \mathcal{M}^c_* := \{M^c_i : i \in [\kappa'], |M^c_i| < (1-\varepsilon)n/k\}.$$

As $|M^c_i| \leq n/k$ for any $i \in [\kappa']$ and $c \in [q]$, we have

$$\frac{(\delta-\nu-3\mu^{1/3})n^2}{k(k-1)} \overset{(3.10),(3.11)}{\leq} |E(\tilde{\mathcal{H}}^c)| = \sum_{i\in[\kappa']} |M^c_i| < \frac{|\mathcal{M}^c_*|(1-\varepsilon)n}{k} + \frac{(\kappa' - |\mathcal{M}^c_*|)n}{k}.$$

This gives

$$|\mathcal{M}^c_*| \leq \frac{(\varepsilon^3/q + 3\mu^{1/3})kn^2}{\varepsilon nk(k-1)} \leq \frac{2\varepsilon^2 n}{q(k-1)}. \tag{3.12}$$

We let

$$\kappa := \min_{c\in[q]}\{|\mathcal{M}^c \setminus \mathcal{M}^c_*|\} = \kappa' - \max_{c\in[q]}\{|\mathcal{M}^c_*|\} = \frac{(\delta-\nu)n \pm 2\varepsilon^2 n/q}{k-1}. \tag{3.13}$$

Thus, by permuting indices, we can assume that for each $c \in [q]$, we have $M^c_1, \ldots, M^c_\kappa \subseteq \mathcal{M}^c \setminus \mathcal{M}^c_*$. For each $(c,i) \in [q] \times [\kappa]$, let

$$F_{c,i} := \bigcup_{j : V(Q^c_j) \in M^c_i} Q^c_j.$$

The fact that $\mathcal{M}^c \setminus \mathcal{M}^c_*$ is a collection of pairwise edge-disjoint matchings of $\tilde{\mathcal{H}}^c \subseteq \mathcal{H}^c$ together with (3.9) implies that, for each $c \in [q]$, the collection $\{F_{c,i} : i \in [\kappa]\}$ consists of pairwise edge-disjoint subgraphs of $G^c_* \subseteq G$, each of which is a union of at least $(1-\varepsilon)n/k$ vertex-disjoint copies of K_k. This with $(\Phi2)_{3.13}$ shows that $(\mathrm{B3})_{3.13}$ holds. As $G^1_*, \dots, G^q_*$ are pairwise edge-disjoint subgraphs, $\{F_{c,i} : (c,i) \in [q] \times [\kappa]\}$ forms a collection of pairwise edge-disjoint subgraphs of G. Thus $(\mathrm{B1})_{3.13}$ holds.

Moreover, for each $c \in [q]$ and each vertex $v \in V(G) \setminus V'$, we have

$$\begin{aligned} |\{i \in [\kappa] : v \in V(F_{c,i})\}| &\geq |\{M \in \{M^c_1, \dots, M^c_\kappa\} : v \in V(M)\}| \\ &\geq |\{M \in \mathcal{M}^c : v \in V(M)\}| - (\kappa' - \kappa) \\ &\geq d_{\tilde{H}^c}(v) - \kappa' + \kappa \overset{(3.11)}{\geq} \kappa - \varepsilon n/q. \end{aligned}$$

Thus $(\mathrm{B2})_{3.13}$ holds. □

3.4. **Graph packing tools.** The following two results from [30] will allow us to pack many bounded degree graphs into appropriate super-regular blow-ups. Lemma 3.14 first allows us to pack graphs into internally regular graphs which still have bounded degree, and Theorem 3.15 allows us to pack the internally regular graphs into an appropriate dense ε-regular graph. The results in [30] are actually significantly more general, mainly because they allow for more general reduced graphs R.

Lemma 3.14. [30, Lemma 7.1] *Suppose $n, \Delta, q, s, k, r \in \mathbb{N}$ with $0 < 1/n \ll \varepsilon \ll 1/s \ll 1/\Delta, 1/k$ and $\varepsilon \ll 1/q \ll 1$ and k divides r. Suppose that $0 < \xi < 1$ is such that $s^{2/3} \leq \xi q$. Let R be a graph on $[r]$ consisting of r/k vertex-disjoint copies of K_k. Let $V_1, \dots, V_r$ be a partition of some vertex set V such that $|V_i| = n$ for all $i \in [r]$. Suppose for each $j \in [s]$, L_j is a graph admitting the vertex partition $(R, X^j_1, \dots, X^j_r)$ such that $\Delta(L_j) \leq \Delta$ and for each $ii' \in E(R)$, we have*

$$\sum_{j=1}^{s} e(L_j[X^j_i, X^j_{i'}]) = (1 - 3\xi \pm \xi)qn,$$

and $|X^j_i| \leq n$. Also suppose that for all $j \in [s]$ and $i \in [r]$, we have sets $W^j_i \subseteq X^j_i$ such that $|W^j_i| \leq \varepsilon n$. Then there exists a graph H on V which is internally q-regular with respect to $(R, V_1, \dots, V_r)$ and a function ϕ which packs $\{L_1, \dots, L_s\}$ into H such that $\phi(X^j_i) \subseteq V_i$, and such that for all distinct $j, j' \in [s]$ and $i \in [r]$, we have $\phi(W^j_i) \cap \phi(W^{j'}_i) = \emptyset$.

Theorem 3.15 (Blow-up lemma for approximate decompositions [30, Theorem 6.1])**.** *Suppose $n,\ q,\ s,\ k, r \in \mathbb{N}$ with $0 < 1/n \ll \varepsilon \ll \alpha, d, d_0, 1/q, 1/k \leq 1$ and $1/n \ll 1/r$ and k divides r. Suppose that R is a graph on $[r]$ consisting of r/k vertex-disjoint copies of K_k. Suppose $s \leq \frac{d}{q}(1 - \alpha/2)n$ and the following hold.*

- $(\mathrm{A1})_{3.15}$ *G is (ε, d)-super-regular with respect to the vertex partition $(R, V_1, \dots, V_r)$.*
- $(\mathrm{A2})_{3.15}$ *$\mathcal{H} = \{H_1, \dots, H_s\}$ is a collection of graphs, where each H_j is internally q-regular with respect to the vertex partition $(R, X_1, \dots, X_k)$, and $|X_i| = |V_i| = n$ for all $i \in [r]$.*
- $(\mathrm{A3})_{3.15}$ *For all $j \in [s]$ and $i \in [r]$, there is a set $W^j_i \subseteq X_i$ with $|W^j_i| \leq \varepsilon n$ and for each $w \in W^j_i$, there is a set $A^j_w \subseteq V_i$ with $|A^j_w| \geq d_0 n$.*
- $(\mathrm{A4})_{3.15}$ *Λ is a graph with $V(\Lambda) \subseteq [s] \times \bigcup_{i=1}^{r} X_i$ and $\Delta(\Lambda) \leq (1 - \alpha)d_0 n$ such that for all $(j, x) \in V(\Lambda)$ and $j' \in [s]$, we have $|\{x' : (j', x') \in N_\Lambda((j, x))\}| \leq q^2$. Moreover, for all $j \in [s]$ and $i \in [r]$, we have $|\{(j, x) \in V(\Lambda) : x \in X_i\}| \leq \varepsilon |X_i|$.*

Then there is a function ϕ packing $\mathcal{H}$ into G such that, writing ϕ_j for the restriction of ϕ to H_j, the following hold for all $j \in [s]$ and $i \in [r]$.

- $(\mathrm{B1})_{3.15}$ *$\phi_j(X_i) = V_i$,*
- $(\mathrm{B2})_{3.15}$ *$\phi_j(w) \in A^j_w$ for all $w \in W^j_i$,*
- $(\mathrm{B3})_{3.15}$ *for all $(j, x)(j', y) \in E(\Lambda)$, we have that $\phi_j(x) \neq \phi_{j'}(y)$.*

3.5. **Miscellaneous.** In the proof of Theorem 1.2, we often partition various graphs into parts with certain properties. The next two lemmas will allow us to obtain such partitions. Lemma 3.16 follows by considering a random equipartition and applying concentration of the hypergeometric distribution. Lemma 3.17 can be proved by assigning each edge of G to $G_1,\dots,G_s$ independently at random according to $(p_1,\dots,p_s)$, and applying Lemma 3.1. We omit the details.

Lemma 3.16. *Suppose $n,T,r\in\mathbb{N}$ with $0<1/n\ll 1/T,1/r\le 1$. Let G be an n-vertex graph. Let $V\subseteq V(G)$ and let $V_1\dots,V_r$ be a partition of V. Then there exists an equipartition $Res_1,\dots,Res_T$ of V such that the following hold.*

(i) *For all $t\in[T]$, $i\in[r]$ and $v\in V(G)$, we have $d_{G,Res_t\cap V_i}(v)=\frac{1}{T}d_{G,V_i}(v)\pm n^{2/3}$,*
(ii) *for all $t\in[T]$, $i\in[r]$, we have $|Res_t\cap V_i|=\frac{1}{T}|V_i|\pm n^{2/3}$.*

Lemma 3.17. *Suppose $n,s\in\mathbb{N}$ with $0<1/n\ll\varepsilon\ll 1/s\le 1$ and $m_i\in[n]$ for each $i\in[2]$. Let G be an n-vertex graph. Suppose that $\mathcal{U}$ is a collection of m_1 subsets of $V(G)$ and $\mathcal{U}'$ is a collection of m_2 pairs of disjoint subsets of $V(G)$ such that each $(U_1,U_2)\in\mathcal{U}'$ satisfies $|U_1|,|U_2|>n^{3/4}$. Let $0\le p_1,\dots,p_s\le 1$ with $\sum_{i=1}^s p_i=1$. Then there exists a decomposition $G_1,\dots,G_s$ of G satisfying the following.*

(i) *For all $i\in[s]$, $U\in\mathcal{U}$ and $v\in V(G)$, we have $d_{G_i,U}(v)=p_id_{G,U}(v)\pm n^{2/3}$,*
(ii) *for all $i\in[s]$ and $(U_1,U_2)\in\mathcal{U}'$ such that $G[U_1,U_2]$ is $(\varepsilon,d_{(U_1,U_2)})$-regular for some $d_{(U_1,U_2)}$, we have that $G_i[U_1,U_2]$ is $(2\varepsilon,p_id_{(U_1,U_2)})$-regular.*

The following lemma allows us to find well-distributed subsets of a collection of large sets. The required sets can be found via a straightforward greedy approach (while avoiding the vertices which would violate (B3)$_{3.18}$ in each step). So we omit the details.

Lemma 3.18. *Suppose $n,s,r\in\mathbb{N}$ and $0<1/n,1/s\ll\varepsilon\ll d<1$. Let A be a set of size n, and for each $(i,j)\in[s]\times[r]$ let $A_{i,j}\subseteq A$ be of size at least dn, and let $m_{i,j}\in\mathbb{N}\cup\{0\}$ be such that for all $i\in[s]$ we have $\sum_{j=1}^r m_{i,j}\le\varepsilon n$. Then there exist sets $B_{1,1},\dots,B_{s,r}$ satisfying the following.*

(B1)$_{3.18}$ *For all $i\in[s]$ and $j\in[r]$, we have $B_{i,j}\subseteq A_{i,j}$ with $|B_{i,j}|=m_{i,j}$,*
(B2)$_{3.18}$ *for all $i\in[s]$ and $j'\neq j''\in[r]$, we have $B_{i,j'}\cap B_{i,j''}=\emptyset$,*
(B3)$_{3.18}$ *for all $v\in A$, we have $|\{(i,j)\in[s]\times[r]:v\in B_{i,j}\}|\le\varepsilon^{1/2}s$.*

The following lemma guarantees a set of k-cliques in a graph G which cover every vertex a prescribed number of times.

Lemma 3.19. *Let $n,m,k,t\in\mathbb{N}$ and $0<1/n\ll 1/t\ll\sigma,1/k<1$ with $k\mid n$. Let G be an n-vertex graph with $\delta(G)\ge(1-\frac{1}{k}+\sigma)n$. Suppose that for each $v\in V(G)$, we have $d_v\in[m]\cup\{0\}$. Then there exists a multi-k-graph H on vertex set $V(G)$ satisfying the following.*

(B1)$_{3.19}$ *For each $e\in E(H)$, we have $G[e]\simeq K_k$,*
(B2)$_{3.19}$ *for each $v\in V(G)$, we have $d_H(v)-d_v=(t+1)m\pm 1$.*

Proof. Let
$$m':=\max_{u,v\in V(G)}\{d_u-d_v\}.$$
Then $m'\in[m]$. For a multi-hypergraph H on vertex set $V(G)$ and $v\in V(G)$, let $p_H(v):=d_H(v)-d_v$. We will prove that for each $\ell\in[m'-1]\cup\{0\}$, there exists a hypergraph H_ℓ satisfying the following.

(H1)$^{\ell}_{3.19}$ For each $e\in E(H)$, we have $G[e]\simeq K_k$,
(H2)$^{\ell}_{3.19}$ $\Delta(H_\ell)\le\ell(t+1)$,
(H3)$^{\ell}_{3.19}$ $\max_{u,v\in V(G)}\{p_{H_\ell}(v)-p_{H_\ell}(u)\}\le m'-\ell$.

Note that $H_0=\emptyset$ satisfies (H1)$^0_{3.19}$–(H3)$^0_{3.19}$. Assume that for some $\ell\in[m'-2]\cup\{0\}$, we have already constructed H_ℓ satisfying (H1)$^{\ell}_{3.19}$–(H3)$^{\ell}_{3.19}$. We will now construct $H_{\ell+1}$.

If $\max_{u\in V(G)}\{p_{H_\ell}(u)\}-\min_{u\in V(G)}\{p_{H_\ell}(u)\}\le 1$, then as $\ell\le m'-2$, we can let $H_{\ell+1}:=H_\ell$, then (H1)$^{\ell+1}_{3.19}$–(H3)$^{\ell+1}_{3.19}$ hold. Thus assume that
$$\max_{u\in V(G)}\{p_{H_\ell}(u)\}-\min_{u\in V(G)}\{p_{H_\ell}(u)\}\ge 2. \tag{3.14}$$

Let

$$A := \{v \in V(G) : p_{H_\ell}(v) > \min_{u \in V(G)} \{p_{H_\ell}(u)\}\} \text{ and } A_{\max} := \{v \in V(G) : p_{H_\ell}(v) = \max_{u \in V(G)} \{p_{H_\ell}(u)\}\}.$$

First assume that $|A| \geq k$. Let $A' \subseteq A$ be a set of at most $k-1$ vertices such that k divides $|A| + |A'|$ and $p_{H_\ell}(v) \geq \max_{u \in A \setminus A'} p_{H_\ell}(u)$ for all $v \in A'$. Note that we have either $A' \subseteq A_{\max}$ or $A_{\max} \subseteq A'$. Then we can take a collection $\mathcal{A} := \{A_1, \dots, A_{t+1}\}$ of (possibly empty) subsets of A such that the following hold for each $i \in [t+1]$.

- $|A_i|$ is divisible by k,
- $|A_i| \leq |A|/t + k$,
- every vertex in A' belongs to exactly two sets in $\mathcal{A}$ and every vertex in $A \setminus A'$ belongs to exactly one set in $\mathcal{A}$.

Now, for each $i \in [t+1]$, we have

$$\delta(G - A_i) \geq \delta(G) - |A_i| \geq (1 - 1/k + \sigma)n - n/t - k \geq (1 - 1/k + \sigma - 2/t)n \geq (1 - 1/k)n.$$

Since $V(G)\setminus A_i$ contains at most n vertices, and $|V(G)\setminus A_i|$ is divisible by k, the Hajnal-Szemerédi theorem implies that there exists a collection $\mathcal{K}_i$ of copies of K_k in G covering all the vertices in $V(G)\setminus A_i$ exactly once. For each $i \in [t+1]$, let $E_i := \{V(K) : K \in \mathcal{K}_i\}$. Then $\bigcup_{i=1}^{t+1} E_i$ covers every vertex in $V(G)\setminus A$ exactly $t+1$ times, while it covers vertices in $A \setminus A'$ exactly t times and vertices in A' exactly $t-1$ times. Let $H_{\ell+1}$ be the multi-k-graph on vertex set $V(G)$ with

$$E(H_{\ell+1}) := H_\ell \cup \bigcup_{i=1}^{t+1} E_i.$$

Then the above construction with $(\mathrm{H1})^{\ell}_{3.19}$ implies $(\mathrm{H1})^{\ell+1}_{3.19}$. Also $(\mathrm{H2})^{\ell}_{3.19}$ implies that $\Delta(H_{\ell+1}) = \Delta(H_\ell) + (t+1) \leq (t+1)(\ell+1)$, thus $(\mathrm{H2})^{\ell+1}_{3.19}$ holds. If $A' \subsetneq A_{\max}$, then every vertex in $A_{\max} \setminus A'$ is covered exactly t times by $\bigcup_{i=1}^{t+1} E_i$. Thus, by (3.14), we have

$$\max_{u \in V(G)} \{p_{H_{\ell+1}}(u)\} = \max_{u \in V(G)} \{p_{H_\ell}(u)\} + t \text{ and } \min_{u \in V(G)} \{p_{H_{\ell+1}}(u)\} = \min_{u \in V(G)} \{p_{H_\ell}(u)\} + t + 1.$$

If $A_{\max} \subseteq A'$, then every vertex in $A_{\max}$ is covered exactly $t-1$ times while every vertex in A is covered either $t-1$ times or t times by $\bigcup_{i=1}^{t+1} E_i$. Thus, by (3.14), we have

$$\max_{u \in V(G)} \{p_{H_{\ell+1}}(u)\} = \max_{u \in V(G)} \{p_{H_\ell}(u)\} + t - 1 \text{ and } \min_{u \in V(G)} \{p_{H_{\ell+1}}(u)\} \geq \min_{u \in V(G)} \{p_{H_\ell}(u)\} + t.$$

In both cases, we have

$$\max_{u,v \in V(G)} \left\{p_{H_{\ell+1}}(u) - p_{H_{\ell+1}}(v)\right\} \leq \max_{u,v \in V(G)} \{p_{H_\ell}(u) - p_{H_\ell}(v)\} - 1 \overset{(\mathrm{H3})^{\ell}_{3.19}}{\leq} m' - \ell - 1.$$

Thus $(\mathrm{H3})^{\ell+1}_{3.19}$ holds.

Next assume that $|A| < k$. Then we take two sets B and C in $V(G)$ such that $B \cap C = A$ and $|B| = |C| = k$. Then similarly as before, we can take two collections E_1 and E_2 of sets of size k such that E_1 covers every vertex in $V(G) \setminus B$ exactly once, and E_2 covers every vertex in $V(G) \setminus C$ exactly once while $G[e] \simeq K_k$ for all $e \in E_1 \cup E_2$. Let $H_{\ell+1}$ be the multi-k-graph with $E(H_{\ell+1}) := H_\ell \cup E_1 \cup E_2$. Then, it is easy to see that both $(\mathrm{H1})^{\ell+1}_{3.19}$ and $(\mathrm{H2})^{\ell+1}_{3.19}$ hold. Also $E_1 \cup E_2$ covers all vertices in $V(G) \setminus A$ exactly once or twice, while it does not cover the vertices in A. Then as before, by using the fact that $\max_{u \in V(G)}\{p_{H_\ell}(u)\} - \min_{u \in V(G)}\{p_{H_\ell}(u)\} \geq 2$, we can show that $(\mathrm{H3})^{\ell+1}_{3.19}$ holds.

Hence, this shows that there exists a hypergraph $H_{m'-1}$ which satisfies $(\mathrm{H1})^{m'-1}_{3.19}$–$(\mathrm{H3})^{m'-1}_{3.19}$. Let $m'' := \max_{v \in V(G)}\{p_{H_{m'-1}}(v)\}$. Then $(\mathrm{H2})^{m'-1}_{3.19}$ implies that $m'' \leq (t+1)m$. Also, by $(\mathrm{H3})^{m'-1}_{3.19}$ every vertex $v \in V(G)$ satisfies $p_{H_{m'-1}}(v) \in \{m''-1, m''\}$. Recall that $\delta(G) \geq (1-1/k)n$ and k divides n. Thus the Hajnal-Szemerédi theorem guarantees a collection E of sets of size k which covers every vertex of G exactly once, while $G[e] \simeq K_k$ for all $e \in E$. Thus, by adding all $e \in E$ to $H_{m'-1}$ exactly $(t+1)m - m''$ times, we obtain a multi-k-graph satisfying $(\mathrm{B1})_{3.19}$ and $(\mathrm{B2})_{3.19}$. $\square$

The following lemma is due to Komlós, Sárközy and Szemerédi [32]. Assertion (B3)$_{3.20}$ is not explicitly stated in [32], but follows immediately from the proof given there (see Section 3.1 in [32]). Given embeddings of graphs H_i and H_j into blown-up k-cliques $Q_i \subseteq G$ and $Q_j \subseteq G$, the 'clique walks' guaranteed by Lemma 3.20 will allow us to find suitable connections between (the images of) H_i and H_j in G.

Lemma 3.20. *Let $r, k \in \mathbb{N} \setminus \{1\}$. Suppose that R is an r-vertex graph with $\delta(R) \geq \left(1 - \frac{1}{k}\right)r + 1$. Suppose that Q_1, Q_2 are two not necessarily disjoint subsets of $V(R)$ of size k such that $Q_1 = \{x_1, \ldots, x_k\}$ and $Q_2 = \{y_1, \ldots, y_k\}$ with $R[Q_1] \simeq K_k$ and $R[Q_2] \simeq K_k$. Then there exists a walk $W = (z_1, \ldots, z_t)$ in R satisfying the following.*

(B1)$_{3.20}$ *$3k \leq t \leq 3k^3$ and $k \mid t$,*
(B2)$_{3.20}$ *for all $i, j \in [t]$ with $|i - j| \leq k - 1$, we have $z_i z_j \in E(R)$,*
(B3)$_{3.20}$ *for each $i \in [k]$, we have $z_i = x_i$ and $z_{t-k+i} = y_i$.*

The following lemma also can be proved using a simple greedy algorithm. We omit the proof.

Lemma 3.21. *Let $\Delta, k, t \in \mathbb{N} \setminus \{1\}$. Let H be a graph with $\Delta(H) \leq \Delta$ and let $X \subseteq V(H)$ be a set with $|X| \geq \Delta^k t$. Then there exists a k-independent set $Y \subseteq X$ of H with $|Y| = t$.*

Lemma 3.22. *Let $r, k, q, s \in \mathbb{N} \setminus \{1\}$ with $0 < 1/r \ll 1/k, 1/q \leq 1$. Let R be an r-vertex graph with $\delta(R) \geq (1 - \frac{1}{k})r$. Let $\mathcal{F}$ be a multi-$(k-1)$-graph on $V(R)$ with $\Delta(\mathcal{F}) \leq q$ and $E(\mathcal{F}) = \{F_1, \ldots, F_s\}$ such that $R[F_i] \simeq K_{k-1}$ for all $i \in [s]$. Then there exists a multi-k-graph $\mathcal{F}^*$ on $V(R)$ with $E(\mathcal{F}^*) = \{F_1^*, \ldots, F_s^*\}$ and such that*

(B1)$_{3.22}$ *$\Delta(\mathcal{F}^*) \leq (k+1)q$,*
(B2)$_{3.22}$ *for all $i \in [s]$, we have $F_i \subseteq F_i^*$ and $R[F_i^*] \simeq K_k$.*

Proof. Since $\mathcal{F}$ is a multi-$(k-1)$-graph, we have $s \leq \Delta(\mathcal{F})r/(k-1) \leq qr$. We consider an auxiliary bipartite graph Aux with vertex partition $(E(\mathcal{F}), V(R) \times [kq])$ such that F_i is adjacent to $(v, j) \in V(R) \times [kq]$ if $v \in N_R(F_i)$. For any set X of $k-1$ vertices in R, we have $d_R(X) \geq r/k$. Thus, any vertex F_i of the graph Aux has degree at least $kqd_R(F_i) \geq kq \cdot (r/k) \geq s = |E(\mathcal{F})|$. Thus, the graph Aux contains a matching M covering every $F_i \in E(\mathcal{F})$. For each $(F_i, (v, j)) \in M$, let $F_i^* := F_i \cup \{v\}$. Then (B2)$_{3.22}$ holds. On the other hand, for any vertex $v \in V(R)$, we have $d_{\mathcal{F}^*}(v) = d_{\mathcal{F}}(v) + |\{j \in [kq] : d_M((v,j)) = 1\}| \leq d_{\mathcal{F}}(v) + kq \leq (k+1)q$. Thus (B1)$_{3.22}$ holds too. □

The final tool we will collect implies that a (k, η)-chromatic η-separable bounded degree graph has a small separator S and a $(k+1)$-colouring in which one colour class is small and only consists of vertices far away from S.

Lemma 3.23. *Suppose that $n, t, \Delta, k \in \mathbb{N}$ and $\Delta \geq 2$. Suppose that H is an η-separable n-vertex graph with $\Delta(H) \leq \Delta$. If H admits a $(k+1)$-colouring with colour classes $W_0, \ldots, W_k$ with $|W_0| \leq \eta n$, then there exists a $\Delta^{t+2}\eta$-separator S of H with $N_H^t(S) \cap W_0 = \emptyset$.*

Proof. As H is η-separable, there exists an η-separator S' of H. Consider $S := (S' \cup N_H^{t+1}(W_0)) \setminus N_H^t(W_0)$. It is obvious that such a choice satisfies $N_H^t(S) \cap W_0 = \emptyset$. Furthermore, as $|W_0| \leq \eta n$ and $\Delta \geq 2$, we have $|S| \leq \Delta^{t+2}\eta n$. Moreover, any component of $H - S$ is either a subset of a a component of $H - S'$ or a subset of $N_H^t(W_0)$. Hence, it has size at most $\Delta^{t+2}\eta n$, and S is a separator as desired. □

4. Constructing an appropriate partition of a separable graph

In Section 6 we will decompose the host graph G into graphs G_t, F_t and F_t' with $t \in [T]$ for some bounded T. We will also construct an exceptional set V_0 and reservoir sets Res_t. We now need to partition each graph $H \in \mathcal{H}$ so that this partition reflects the above decomposition of G. This will enable us to apply the blow-up lemma for approximate decompositions (Theorem 3.15) in Section 5. The next lemma ensures that we can prepare each graph $H \in \mathcal{H}$ in an appropriate manner. It gives a partition of $V(H)$ into X, Y, Z, A. Later we will aim to embed the vertices in A into V_0, and vertices in $Y \cup Z$ will be embedded into Res_t using Lemma 3.6. Most of the vertices in X will be embedded into a super-regular blown-up K_k-factor in G_t via Theorem 3.15, while the remaining vertices of X will be embedded into Res_t. The set Z will contain a suitable separator H_0 of H. The neighbourhoods of the exceptional vertices $a_\ell \in A$ will be allocated to Y. Moreover, (A2)$_{4.1}$ and (A3)$_{4.1}$ ensure that we allocate them to sets corresponding to (evenly distributed) cliques of R—the latter enables us to satisfy the second part of (B3)$_{4.1}$.

Lemma 4.1. *Suppose $n, m, r, k, h, \Delta \in \mathbb{N}$ with $0 < 1/n \ll \eta \ll \varepsilon \ll 1/h \ll 1/k, \sigma, 1/\Delta < 1$ and $0 < \eta \ll 1/r < 1$ such that $k \mid r$. Let H be an n-vertex (k, η)-chromatic graph with $e(H) = m$ and $\Delta(H) \le \Delta$. Let R and Q be graphs with $V(R) = V(Q) = [r]$ such that Q is a union of r/k vertex-disjoint copies of K_k. For $n' \in [\varepsilon n]$, let $C_1, \dots, C_{n'}$ be subsets of $[r]$ of size $k-1$, and $C^*_1, \dots, C^*_{n'}$ be subsets of $[r]$ of size k. Let $\mathcal{F}$ and $\mathcal{F}^*$ be multi-hypergraphs on $[r]$ with $E(\mathcal{F}) = \{C_1, \dots, C_{n'}\}$ and $E(\mathcal{F}^*) = \{C^*_1, \dots, C^*_{n'}\}$. Suppose that $n_1, \dots, n_r$ are integers. Suppose the following hold.*

(A1)$_{4.1}$ $\delta(R) \ge (1 - \frac{1}{k} + \sigma)r$,
(A2)$_{4.1}$ *for each $\ell \in [n']$, we have $C_\ell \subseteq C^*_\ell$ and $R[C^*_\ell] \simeq K_k$,*
(A3)$_{4.1}$ $\Delta(\mathcal{F}^*) \le \varepsilon^{2/3} n/r$,
(A4)$_{4.1}$ *for each $i \in [r]$, we have $n_i = (1 \pm \varepsilon^{1/2})n/r$, and $n' + \sum_{i \in [r]} n_i = n$.*

Then there exists a randomised algorithm which always returns an ordered partition $(X_1, \dots, X_r, Y_1, \dots, Y_r, Z_1, \dots, Z_r, A)$ of $V(H)$ such that $A = \{a_1, \dots, a_{n'}\}$ is a 3-independent set of H and the following hold, where $X := \bigcup_{i \in [r]} X_i$, $Y := \bigcup_{i \in [r]} Y_i$, and $Z := \bigcup_{i \in [r]} Z_i$.

(B1)$_{4.1}$ *For each $\ell \in [n']$, we have $d_H(a_\ell) \le \frac{2(1+1/h)m}{n}$,*
(B2)$_{4.1}$ *for each $\ell \in [n']$, we have $N_H(a_\ell) \subseteq \bigcup_{i \in C_\ell} Y_i \setminus N^1_H(Z)$,*
(B3)$_{4.1}$ *$H[X]$ admits the vertex partition $(Q, X_1, \dots, X_r)$, and $H \setminus E(H[X])$ admits the vertex partition $(R, X_1 \cup Y_1 \cup Z_1, \dots, X_r \cup Y_r \cup Z_r)$,*
(B4)$_{4.1}$ *for each $ij \in E(Q)$, we have $e_H(X_i, X_j) = \frac{2m \pm \varepsilon^{1/5} n}{(k-1)r}$,*
(B5)$_{4.1}$ *for each $i \in [r]$, we have $|X_i| + |Y_i| + |Z_i| = n_i \pm \eta^{1/4} n$ and $|Y_i| \le 2\varepsilon^{1/3} n/r$,*
(B6)$_{4.1}$ *$N^1_H(X) \setminus X \subseteq Z$ and $|Z| \le 4\Delta^{3k^3} \eta^{0.9} n$.*

Moreover, the algorithm has the following additional property, where the expectation is with respect to all possible outputs.

(B7)$_{4.1}$ *For all $\ell \in [n']$ and $i \in C_\ell$, we have $\mathbb{E}[N_H(a_\ell) \cap Y_i] \le \frac{2(1+1/h)m}{(k-1)n}$.*

(B1)$_{4.1}$ and (B7)$_{4.1}$ ensure that each embedding of some H in G does not use too many edges incident to the exceptional set V_0.

Proof. Write $r' := r/k$ and $Q = \bigcup_{s=1}^{r'} Q_s$, where each Q_s is a copy of K_k, and let $\binom{R}{K_k} = \{Q'_1, \dots, Q'_q\}$ be the collection of all copies of K_k in R. By permuting indices if necessary, we may assume that $V(Q'_1) = \{1, \dots, k\}$. Note that $q \le r^k$. As Q is a K_k-factor on $[r]$, for each $i \in [r]$, there exists a unique $j \in [r']$ such that $i \in Q_j$. For all $s \in [r']$, $s' \in [q]$ and $k' \in [k]$, we define $q_s(k'), q'_{s'}(k') \in [r]$ to be the k'-th smallest number in $V(Q_s)$ and $V(Q'_{s'})$ respectively. Thus

$$V(Q_s) = \{q_s(1), \dots, q_s(k)\} \text{ and } V(Q'_{s'}) = \{q'_{s'}(1), \dots, q'_{s'}(k)\}.$$

For all $s \in [q]$ and $k' \in [k]$, let

$$Q'_{s,k'} := Q'_s \setminus \{q'_s(k')\} \quad \text{and} \quad d_{s,k'} := |\{\ell \in [n'] : C^*_\ell = V(Q'_s) \text{ and } C_\ell = V(Q'_{s,k'})\}|. \tag{4.1}$$

Note that for each $i \in [r]$ we have

$$\sum_{s \in [q] : i \in V(Q'_s)} \sum_{k' \in [k]} d_{s,k'} = d_{\mathcal{F}^*}(i) \quad \text{and} \quad \sum_{(s,k') \in [q] \times [k]} d_{s,k'} = n'. \tag{4.2}$$

Our strategy is as follows. Consider a $(k+1)$-colouring $(W_0, \dots, W_k)$ of H with $|W_0| \le \eta n$ and an $\Delta^{3k^3+3}\eta n$-separator S of H guaranteed by Lemma 3.23 (applied with $t = 3k^3 + 1$). Thus we can partition the k-chromatic graph $H \setminus W_0$ into $H_0, \dots, H_t$ such that each $H_{t'}$ is small, there are no edges between $H_{t'}$ and $H_{t''}$ whenever $0 \notin \{t', t''\}$ and $V(H_0) = S$. We will distribute the vertices of each graph $H_{t'}$ into $\bigcup_{i \in V(Q_s)} X_i$ or $\bigcup_{i \in V(Q'_s)} (Y_i \cup Z_i)$ for an appropriate s. In particular, $V(H_0)$ will be allocated to $\bigcup_{i\in V(Q'_1)} Z_i = \bigcup_{i\in[k]} Z_i$. As Q'_s and Q_s are copies of K_k in R and Q, respectively, and as $H_{t'}$ is k-chromatic, this would allow us to achieve (B3)$_{4.1}$ if we ignore the edges incident to $V(H_0)\cup W_0$. In Steps 5 and 6 we will use 'clique walks' obtained from Lemma 3.20 to connect up the $H_{t'}$ with H_0 in a way which respects the colour classes of $H\setminus W_0$. We can thus allocate the vertices in $N_H^{3k^3}(V(H_0))$ in a way that will satisfy (B3)$_{4.1}$. Finally, we will allocate the vertices in W_0. As W_0 is far from $V(H_0)$, each vertex in W_0 only has its neighbours in a single $H_{t'}$, hence it will be simple to assign each vertex in W_0 to some Z_i with $i\in[r]$ according to where the vertices of $H_{t'}$ are assigned.

Step 1. Separating H. As H is (k,η)-chromatic, applying Lemma 3.23 with $t = 3k^3+1$ implies that there exists a partition $(W_0, W_1,\dots,W_k)$ of $V(H)$ into independent sets and an $\eta^{0.9}$-separator S such that

$$|S|, |W_0| \le \eta^{0.9} n \text{ and } W_0 \cap N_H^{3k^3+1}(S) = \emptyset. \tag{4.3}$$

Since S is an $\eta^{0.9}$-separator of H, it follows that there exists a partition $S =: \widetilde{V}_0,\dots,\widetilde{V}_t$ of $V(H)$ such that the following hold, where $V_{t'} := \widetilde{V}_{t'}\setminus W_0$ and $H_{t'} := H[V_{t'}]$ for each $t'\in[t]\cup\{0\}$.

(H1)$_{4.1}$ $\eta^{-0.9}/2 \le t \le 2\eta^{-0.9}$,
(H2)$_{4.1}$ $\eta^{0.9}n/2 \le |V_{t'}| \le 2\eta^{0.9}n$ for $t'\in[t]$,
(H3)$_{4.1}$ for $t'\neq t''\in[t]$, we have that $E_H(\widetilde{V}_{t'},\widetilde{V}_{t''})=\emptyset$, and $m-2\Delta\eta^{0.9}n \le \sum_{t'\in[t]} e(H_{t'}) \le m$.

Indeed, as S is an $\eta^{0.9}$-separator of H, $H\setminus S$ only consists of components of size at most $\eta^{0.9}n$. By letting $\tilde{V}_0 := S$ (and thus $V_0 = S$) and letting each of $\widetilde{V}_1,\dots,\widetilde{V}_t$ be appropriate unions of components of $H\setminus S$, we can ensure that both (H1)$_{4.1}$ and (H2)$_{4.1}$ hold. By the construction, the first part of (H3)$_{4.1}$ holds too. Since there are at most $\Delta(H)|S\cup W_0| \le 2\Delta\eta^{0.9}n$ edges which are incident to some vertex in $W_0\cup V_0$, the second part of (H3)$_{4.1}$ holds as well.

For each $t'\in[t]\cup\{0\}$ and $k'\in[k]$, we let

$$W_{k'}^{t'} := V_{t'}\cap W_{k'}.$$

Step 2. Choosing the exceptional set A. Let

$$L := \{x\in V(H) : d_H(x) \le \frac{2(1+1/h)m}{n}\}.$$

L contains the 'low degree' vertices within which we will choose A in order to satisfy (B1)$_{4.1}$. Note that $2m = \sum_{x\in V(H)} d_H(x) \ge \frac{2(1+1/h)m}{n}(n-|L|)$, thus

$$|L| \ge n/(2h). \tag{4.4}$$

For each $t'\in[t]$, let $k(t')\in[k]$ be an index such that

$$|L\cap W_{k(t')}^{t'}| \ge \frac{1}{k}|L\cap V(H_{t'})|. \tag{4.5}$$

Such a number $k(t')$ exists as $W_1^{t'},\dots,W_k^{t'}$ forms a partition of $V_{t'} = V(H_{t'})$.

Now, we choose a partition $\mathcal{H},\mathcal{H}'_{1,1},\dots,\mathcal{H}'_{1,k},\mathcal{H}'_{2,1},\dots,\mathcal{H}'_{q,k}$ of $\{H_1,\dots,H_t\}$ satisfying the following for each $(s,k')\in[q]\times[k]$.

(H4)$_{4.1}$ $v(\mathcal{H}'_{s,k'}) = \varepsilon^{-1/10}d_{s,k'} + 2k\eta^{2/5}n \pm \eta^{2/5}n$ and
$\sum_{t':H_{t'}\in\mathcal{H}'_{s,k'}} |V(H_{t'})\cap L| \ge \varepsilon^{-1/11}d_{s,k'} + \eta^{1/2}n$.

We will choose A within the vertex sets of the graphs in $\mathcal{H}'_{1,1},\dots,\mathcal{H}'_{q,k}$. Moreover, we will allocate all the other vertices of the graphs in each $\mathcal{H}'_{s,k'}$ to $Y\cup Z$.

Claim 1. *There exists a partition $\mathcal{H},\mathcal{H}'_{1,1},\dots,\mathcal{H}'_{1,k},\mathcal{H}'_{2,1},\dots,\mathcal{H}'_{q,k}$ of $\{H_1,\dots,H_t\}$ satisfying* (H4)$_{4.1}$.

Proof. For each $t' \in [t]$, we choose $i_{t'}$ independently at random from $[q] \times [k] \cup \{(0,0)\}$ such that for each $(s,k') \in [q] \times [k]$ we have

$$\mathbb{P}[i_{t'} = (s,k')] = \frac{\varepsilon^{-1/10} d_{s,k'}}{n} + 2k\eta^{2/5} \quad \text{and} \quad \mathbb{P}[i_{t'} = (0,0)] = 1 - \frac{\varepsilon^{-1/10} n'}{n} - 2qk^2\eta^{2/5}.$$

An easy calculation based on (4.2) shows that this defines a probability distribution. For each $(s,k') \in [q] \times [k]$, we let

$$\mathcal{H} := \{H_{t'} : t' \in [t], i_{t'} = (0,0)\} \quad \text{and} \quad \mathcal{H}'_{s,k'} := \{H_{t'} : t' \in [t], i_{t'} = (s,k')\}.$$

Then it is easy to combine a Chernoff bound (Lemma 3.1) with $(\mathrm{H1})_{4.1}$, $(\mathrm{H2})_{4.1}$, (4.4) and the fact that $|V(H)| = n$ to check that the resulting partition satisfies $(\mathrm{H4})_{4.1}$ with positive probability. This proves the claim. □

By permuting indices on $[t]$, we may assume that for some $t_* \in [t]$, we have

$$\mathcal{H} = \{H_1, \dots, H_{t_*}\} \quad \text{and} \quad \bigcup_{(s,k') \in [q] \times [k]} \mathcal{H}'_{s,k'} = \{H_{t_*+1}, \dots, H_t\}.$$

For each $(s,k') \in [q] \times [k]$, let

$$L_{s,k'} := \bigcup_{t' : H_{t'} \in \mathcal{H}'_{s,k'}} (L \cap W^{t'}_{k(t')}) \setminus N^{3k^3+2}_H(V_0 \cup W_0). \tag{4.6}$$

Then by (4.3) and (4.5) we have

$$|L_{s,k'}| \geq \sum_{t' : H_{t'} \in \mathcal{H}'_{s,k'}} \frac{1}{k} |L \cap V(H_{t'})| - 8\Delta^{3k^3+2}\eta^{0.9} n \overset{(\mathrm{H4})_{4.1}}{\geq} \varepsilon^{-1/11} d_{s,k'}/k + \eta^{1/2} n/(2k) \geq \Delta^3 d_{s,k'}.$$

For each $(s,k') \in [q] \times [k]$, we apply Lemma 3.21 to $L_{s,k'}$ to obtain a subset of $L_{s,k'}$ with size exactly $d_{s,k'}$ which is 3-independent in H. Write this 3-independent set as

$$\{a_\ell : \ell \in [n'], C^*_\ell = V(Q'_s) \text{ and } C_\ell = V(Q'_{s,k'})\}. \tag{4.7}$$

This is possible by (4.1) and (4.2) and defines vertices $a_1, \dots, a_{n'}$. Let $A := \{a_1, \dots, a_{n'}\}$. By (4.6) and $(\mathrm{H3})_{4.1}$, A is still a 3-independent set in H. As $a_\ell \in L$, we know that

$$d_H(a_\ell) \leq 2(1 + 1/h)m/n. \tag{4.8}$$

Moreover, for $\ell \in [n']$ and $t' \in [t]$, we have the following.

If $a_\ell \in V_{t'}$, then $t' \in [t] \setminus [t_*]$ and $a_\ell \in W^{t'}_{k(t')} \setminus N^{3k^3+2}_H(V_0 \cup W_0)$. (4.9)

In particular, we have $N_H(a_\ell) \cap N^{3k^3+1}_H(V_0 \cup W_0) = \emptyset$. Thus if $a_\ell \in V_{t'}$, then

$$N_H(a_\ell) \subseteq \bigcup_{k'' \in [k] \setminus \{k(t')\}} W^{t'}_{k''} \setminus N^{3k^3+1}_H(V_0 \cup W_0). \tag{4.10}$$

Step 3. Allocating the neighbourhood of A. We will allocate $N_H(A)$ to Y. We will achieve this by suitably allocating $V(\mathcal{H}'_{s,k'})$ for each $(s,k') \in [q] \times [k]$. This will allocate $N_H(A)$ via (4.10). Note that all choices until now are deterministic. Next we run the following random procedure.

For each $t' \in [t] \setminus [t_*]$, let $(s,k') \in [q] \times [k]$ be such that $H_{t'} \in \mathcal{H}'_{s,k'}$, and choose a permutation $\pi_{t'}$ on $[k]$ independently and uniformly at random among all permutations such that $\pi_{t'}(k') = k(t')$. (4.11)

(Note that this is the only place that our choice is random.) Thus one value of $\pi_{t'}$ is fixed, while all other $k-1$ values are chosen at random. We choose $\pi_{t'}$ in this way because we wish to distribute $N_H(a_\ell)$ to $\bigcup_{i \in C_\ell} Y_i$, so that later $(\mathrm{B2})_{4.1}$ is satisfied. Setting $\pi_{t'}(k') = k(t')$ will ensure that no vertex in $N_H(a_\ell)$ will be distributed to Y_i with $i \in C^*_\ell \setminus C_\ell$. Moreover, as $\pi_{t'}$ is chosen uniformly at random, $N_H(a_\ell)$ will be distributed to $\bigcup_{i \in C_\ell} Y_i$ in a uniform way, which will guarantee that $(\mathrm{B7})_{4.1}$ holds.

Indeed, for $\ell \in [n'], (s,k') \in [q] \times [k]$ and $t' \in [t] \setminus [t_*]$ such that $a_\ell \in L_{s,k'} \cap V_{t'}$, and for any $k'' \in [k] \setminus \{k'\}$, the number $\pi_{t'}(k'')$ is chosen uniformly at random among $[k] \setminus \{k(t')\}$, thus we have

$$\mathbb{E}[|N_H(a_\ell) \cap W^{t'}_{\pi_{t'}(k'')}|] \leq \frac{d_H(a_\ell)}{k-1} \overset{(4.8)}{\leq} \frac{2(1+1/h)m}{(k-1)n}. \tag{4.12}$$

For each $i \in [r]$, let

$$\tilde{Y}_i := \bigcup_{(s,k'):i=q'_s(k')} \bigcup_{k''\in[k]} \bigcup_{H_{t'}\in\mathcal{H}'_{s,k''}} W^{t'}_{\pi_{t'}(k')} \setminus A \quad \text{and} \quad \tilde{Y} := \bigcup_{i\in[r]} \tilde{Y}_i. \tag{4.13}$$

Step 4. Allocating the remaining vertices to X and Y. Later the vertices in $\tilde{Y}_i$ will be assigned to Y_i (except those which are too close to V_0 in H, which will be assigned to Z). The sizes of the sets X_i will be almost identical. (Note that because of (B3)$_{4.1}$, it is not possible to prescribe different sizes for X_i and X_j if i and j lie in the same copy of K_k in Q.) Thus, in order to ensure (B5)$_{4.1}$, we need to decide how many more vertices other than $\tilde{Y}_i$ we will assign to the set Y_i. As part of this we now decide which of the $H_{t'} \in \mathcal{H}$ are allocated to X and which are allocated to Y (again, vertices close to V_0 will be assigned to Z). Note that we have

$$\begin{aligned} |\tilde{Y}_i| &\leq \sum_{(s,k'):i=q'_s(k')} \sum_{k''\in[k]} \sum_{H_{t'}\in\mathcal{H}'_{s,k''}} |H_{t'}| \overset{(\mathrm{H4})_{4.1}}{\leq} \sum_{s:i\in V(Q'_s)} \sum_{k''\in[k]} (\varepsilon^{-1/10} d_{s,k''} + 3k\eta^{2/5} n) \\ &\overset{(4.2)}{\leq} \varepsilon^{-1/10} d_{\mathcal{F}^*}(i) + 3k^2 q\eta^{2/5} n \overset{(\mathrm{A3})_{4.1}}{\leq} \varepsilon^{1/2} n/r. \end{aligned} \tag{4.14}$$

For each $i \in [r]$, let $\tilde{n} := (1-2\varepsilon^{1/2})n/r$, and

$$\tilde{n}_i := n_i - \tilde{n} - |\tilde{Y}_i| \overset{(\mathrm{A4})_{4.1}}{\leq} \frac{\varepsilon^{1/3} n}{(h+1)r}, \text{ then } \tilde{n}_i \overset{(\mathrm{A4})_{4.1}}{\geq} \varepsilon^{1/2} n/r - |\tilde{Y}_i| \overset{(4.14)}{\geq} 0. \tag{4.15}$$

By applying Lemma 3.19 with $R, h, \sigma, \varepsilon^{1/3}n/((h+1)r)$ and $\tilde{n}_i$ playing the roles of G, t, σ, m and d_v, respectively, we obtain a multi-k-graph $\mathcal{F}^\#$ on $[r]$ such that for each $Q \in E(\mathcal{F}^\#)$, we have $R[Q] \simeq K_k$, and

$$\textit{for each } i \in [r], \textit{ we have } d_{\mathcal{F}^\#}(i) = \tilde{n}_i + \tfrac{\varepsilon^{1/3} n}{r} \pm 1. \tag{4.16}$$

This implies

$$\begin{aligned} N &:= \sum_{i\in[r]} (\tilde{n} - \frac{\varepsilon^{1/3} n}{r} + d_{\mathcal{F}^\#}(i)) - |V_0 \cup W_0| \overset{(4.15)}{=} \sum_{i\in[r]} (n_i - |\tilde{Y}_i| \pm 1) - |V_0 \cup W_0| \\ &\overset{(\mathrm{A4})_{4.1}}{=} n - n' - |\tilde{Y}| - |V_0 \cup W_0| \pm r. \end{aligned} \tag{4.17}$$

Note that we have

$$v(\mathcal{H}) = |V(H) \setminus (\tilde{Y} \cup A \cup V_0 \cup W_0)| = N \pm r. \tag{4.18}$$

Our target is to assign roughly $d_{\mathcal{F}^\#}(i)$ extra vertices to Y_i in addition to $\tilde{Y}_i$, and assign roughly $\tilde{n} - \frac{\varepsilon^{1/3} n}{r}$ vertices to X_i, and a negligible amount of vertices to Z_i. Then $|X_i| + |Y_i| + |Z_i|$ will be close to n_i as required in (B5)$_{4.1}$.

To achieve this, we partition $\mathcal{H} = \{H_1, \dots, H_{t_*}\}$ into $\mathcal{H}_1, \dots, \mathcal{H}_{r'}, \mathcal{H}^\#_1, \dots, \mathcal{H}^\#_q$ satisfying the following for all $i \in [r']$ and $s \in [q]$.

(H5)$_{4.1}$ $v(\mathcal{H}_i) = k\tilde{n} - \dfrac{k\varepsilon^{1/3} n}{r} \pm \eta^{2/5} n$ and $e(\mathcal{H}_i) = \dfrac{k(m \pm \varepsilon^{2/7} n)}{r}$,

(H6)$_{4.1}$ $v(\mathcal{H}^\#_s) = k \cdot \mathrm{mult}_{\mathcal{F}^\#}(V(Q'_s)) \pm \eta^{2/5} n$.

(Recall that $\mathrm{mult}_{\mathcal{F}^\#}(V(Q'_s))$ denotes the multiplicity of the edge $V(Q'_s)$ in $\mathcal{F}^\#$.) Indeed, such a partition exists by the following claim.

Claim 2. *There exists a partition $\mathcal{H}_1, \dots, \mathcal{H}_{r'}, \mathcal{H}^\#_1, \dots, \mathcal{H}^\#_q$ of $\{H_1, \dots, H_{t_*}\}$ satisfying (H5)$_{4.1}$–(H6)$_{4.1}$.*

Proof. For each $t' \in [t_*]$, we choose $i_{t'}$ independently at random from $\{(0,1),\dots,(0,r'),(1,1),\dots,(1,q)\}$ such that for each $i \in [r']$ and $s \in [q]$:

$$\mathbb{P}[i_{t'} = (0,i)] = \frac{k\tilde{n} - \frac{k\varepsilon^{1/3}n}{r} - \frac{k|V_0\cup W_0|}{r}}{N} \quad \text{and} \quad \mathbb{P}[i_{t'} = (1,s)] = \frac{k\cdot \mathrm{mult}_{\mathcal{F}^\#}(V(Q'_s))}{N}.$$

Since $\sum_{s\in[q]} k\cdot \mathrm{mult}_{\mathcal{F}^\#}(V(Q'_s)) = k|E(\mathcal{F}^\#)| = \sum_{i\in[r]} d_{\mathcal{F}^\#}(i)$, an easy calculation based on (4.17) shows that this defines a probability distribution. For all $i \in [r']$ and $s \in [q]$, we let

$$\mathcal{H}_i := \{H_{t'} : t' \in [t_*], i_{t'} = (0,i)\} \quad \text{and} \quad \mathcal{H}^\#_s := \{H_{t'} : t' \in [t_*], i_{t'} = (1,s)\}.$$

Then it is easy to combine a Chernoff bound (Lemma 3.1) with (H1)$_{4.1}$, (H2)$_{4.1}$ and (4.18) to check that the resulting partition satisfies (H5)$_{4.1}$ and (H6)$_{4.1}$ with positive probability. This proves the claim. □

By permuting indices on $[t_*]$, we may assume that for some $t^* \in [t_*]$ we have

$$\bigcup_{i\in[r']} \mathcal{H}_i = \{H_1,\dots,H_{t^*}\} \quad \text{and} \quad \bigcup_{s\in[q]} \mathcal{H}^\#_s = \{H_{t^*+1},\dots,H_{t_*}\}.$$

In order to obtain (B3)$_{4.1}$–(B5)$_{4.1}$, we need to distribute vertices of the graphs in $\mathcal{H}_i$ into $\{X_j : j \in V(Q_i)\}$ and vertices of the graphs in $\mathcal{H}^\#_s$ into $\{Y_j : j \in V(Q'_s)\}$ so that the resulting vertex sets and edge sets are evenly balanced. For this, we define a permutation $\pi_{t'}$ on $[k]$ for each $t' \in [t_*]$ which will determine how we will distribute these vertices. We will choose these permutations $\pi_1,\dots,\pi_{t_*}$ such that the following hold for all $i \in [r']$, $s \in [q]$ and $k' \neq k'' \in [k]$.

(H7)$_{4.1}$ $\displaystyle\sum_{t':H_{t'}\in\mathcal{H}_i} |W^{t'}_{\pi_{t'}(k')}| = \tilde{n} - \frac{\varepsilon^{1/3}n}{r} \pm \eta^{2/5}n$ and $\displaystyle\sum_{t':H_{t'}\in\mathcal{H}_i} |E_H(W^{t'}_{\pi_{t'}(k')}, W^{t'}_{\pi_{t'}(k'')})| = \frac{2m \pm \varepsilon^{1/4}n}{(k-1)r}$,

(H8)$_{4.1}$ $\displaystyle\sum_{t':H_{t'}\in\mathcal{H}^\#_s} |W^{t'}_{\pi_{t'}(k')}| = \mathrm{mult}_{\mathcal{F}^\#}(V(Q'_s)) \pm \eta^{2/5}n.$

To see that such permutations exist we consider for each $t' \in [t_*]$ a permutation $\pi_{t'} : [k] \to [k]$ chosen independently and uniformly at random. Then, by a Chernoff bound (Lemma 3.1) combined with (H1)$_{4.1}$ and (H2)$_{4.1}$, it is easy to check that $\pi_1,\dots,\pi_{t_*}$ satisfy (H7)$_{4.1}$ and (H8)$_{4.1}$ with positive probability.

Step 5. Clique walks. Recall that V_0 is a separator of both H and $H \setminus W_0$. The vertices in V_0 will be allocated to the sets $Z_1,\dots,Z_k$ which initially correspond to the clique $Q'_1 \subseteq R$ (recall that $V(Q'_1) = \{1,\dots,k\}$). We now identify an underlying structure in R that will be used in Step 6 to ensure that while allocating $V(H)\setminus(V_0 \cup W_0 \cup A)$ to X, Y and Z, we do not violate the vertex partition admitted by R (c.f. (B3)$_{4.1}$). (This is a particular issue when considering edges between separator vertices and the rest of the partition.)

To illustrate this, let $s \in S$ be a separator vertex allocated to $Z_{k'}$. Let x be some vertex in some $H_{t'}$ with $xs \in E(H)$. Suppose $H_{t'}$ is assigned to some clique $Q_i \subseteq Q$ and that this would assign x to some set $X_{i'}$, where $i' \in V(Q_i)$. Furthermore, suppose $i'k'$ is not an edge in R. We cannot simply reassign x to another set X_j to obey the vertex partition admitted by R without also considering the neighbourhood of x in $H_{t'}$. To resolve this, we apply Lemma 3.20 to obtain a suitable 'clique walk' P between Q'_1 and Q_i, i.e. the initial sement of P is $V(Q'_1)$, its final segment is $V(Q_i)$ and each segment of k consecutive vertices in P corresponds to a k-clique in R. We initially assign x to a set $Z_{k''}$ for some $k'' \in [k] \setminus \{k'\}$. We then assign the vertices which are close to x to some $Z_{k'''}$, where the choice of $k''' \in [r]$ is determined by P. (In order to connect Y to V_0, we also choose similar clique walks starting with Q'_1 and ending with Q'_s for each $s \in [q]$.)

To define the clique walks formally, for each $t' \in [t]$, let

$$P_{t'} := \begin{cases} Q_i & \text{if } H_{t'} \in \mathcal{H}_i \text{ for some } i \in [r'], \\ Q'_s & \text{if } H_{t'} \in \mathcal{H}^\#_s \text{ for some } s \in [q], \\ Q'_s & \text{if } H_{t'} \in \mathcal{H}'_{s,k'} \text{ for some } (s,k') \in [q]\times[k], \end{cases} \quad \text{and} \quad \begin{array}{l} \{p_{t'}(1),\dots,p_{t'}(k)\} := P_{t'}, \\ \text{where } p_{t'}(1) < \dots < p_{t'}(k). \end{array} \tag{4.19}$$

By using (A1)$_{4.1}$, we can apply Lemma 3.20 for each $t' \in [t]$ with $V(Q'_1)$ and $V(P_{t'})$ playing the roles of Q_1 and Q_2 in order to obtain a walk $j(t',1),\dots,j(t',b_{t'}k)$ in R such that

$$\text{for all distinct } i,i' \in [b_{t'}k] \text{ with } |i-i'| \leq k-1, \text{ we have } j(t',i)j(t',i') \in E(R), \text{ and for each } k' \in [k] \text{ we have } j(t',k') = \pi_{t'}(k') \text{ and } j(t',(b_{t'}-1)k+k') = p_{t'}(k'). \tag{4.20}$$

Moreover, for each $t' \in [t]$, we have

$$3 \leq b_{t'} \leq 3k^2. \tag{4.21}$$

As described above we will later distribute some vertices of $V_{t'} \cap N^{(b_{t'}-1)k}(V_0)$ to $\bigcup_{k' \in [(b_{t'}-1)k]} Z_{j(t',k')}$ so that we can ensure (B3)$_{4.1}$ and (B6)$_{4.1}$ hold.

Step 6. Iterative construction of the partition. Now, we will distribute the vertices of each $H_{t'}$ into $X_1,\dots,X_r,Y_1,\dots,Y_r,Z_1,\dots,Z_r$ in such a way that (B1)$_{4.1}$–(B7)$_{4.1}$ hold. (In particular, as discussed earlier, we will have $\widetilde{Y}_i \subseteq Y_i$.) To achieve this, for each $t' = 0,1,\dots,t$, we iteratively define sets $X_1^{t'},\dots,X_r^{t'}$, $Y_1^{t'},\dots,Y_r^{t'}$, $Z_1^{t'},\dots,Z_r^{t'}$. First, for each $k' \in [k]$, let $Z_{k'}^0 := W_{k'}^0$ and for all $i \in [r]$ and $i' \in [r] \setminus [k]$, let

$$X_i^0 := \emptyset, \;\; Y_i^0 := \emptyset \;\text{ and }\; Z_{i'}^0 := \emptyset.$$

We will write

$$V^{t'} := \bigcup_{t''=0}^{t'} V_{t''}, \quad X^{t'} := \bigcup_{i\in[r]} X_i^{t'}, \quad Y^{t'} := \bigcup_{i\in[r]} Y_i^{t'} \quad \text{and} \quad Z^{t'} := \bigcup_{i\in[r]} Z_i^{t'}.$$

Assume that for some $t' \in [t]$, we have already defined a partition $X_1^{t'-1},\dots,X_r^{t'-1}, Y_1^{t'-1},\dots,Y_r^{t'-1}, Z_1^{t'-1},\dots,Z_r^{t'-1}$ of $V^{t'-1}$ satisfying the following.

(Z1)$^{t'-1}_{4.1}$ For all $i' \in [r']$ and $i \in V(Q_{i'})$, let k' be so that $i = q_{i'}(k')$. Then we have
$$\bigcup_{t''\in[t'-1]:H_{t''}\in\mathcal{H}_{i'}} W^{t''}_{\pi_{t''}(k')} \setminus N_H^{(b_{t''}-1)k}(V_0) \subseteq X_i^{t'-1} \subseteq \bigcup_{t''\in[t'-1]:H_{t''}\in\mathcal{H}_{i'}} W^{t''}_{\pi_{t''}(k')},$$

(Z2)$^{t'-1}_{4.1}$ for each $i \in [r]$, we have
$$\bigcup_{k'\in[k]} \bigcup_{\substack{t''\in[t'-1]\setminus[t^*]:\\ p_{t''}(k')=i}} W^{t''}_{\pi_{t''}(k')} \setminus N_H^{(b_{t''}-1)k}(V_0) \subseteq Y_i^{t'-1} \subseteq \bigcup_{k'\in[k]} \bigcup_{\substack{t''\in[t'-1]\setminus[t^*]:\\ p_{t''}(k')=i}} W^{t''}_{\pi_{t''}(k')},$$

(Z3)$^{t'-1}_{4.1}$ for all $ij \notin E(Q)$, we have $e_H(X_i^{t'-1}, X_j^{t'-1}) = 0$,

(Z4)$^{t'-1}_{4.1}$ for all $ij \notin E(R)$, we have $e_H(X_i^{t'-1}, Z_j^{t'-1}) = e_H(Y_i^{t'-1}, Z_j^{t'-1}) = e_H(Y_i^{t'-1}, Y_j^{t'-1}) = e_H(Z_i^{t'-1}, Z_j^{t'-1}) = 0$,

(Z5)$^{t'-1}_{4.1}$ $N_H^1(X^{t'-1}) \setminus X^{t'-1} \subseteq Z^{t'-1} \subseteq N_H^{3k^3}(V_0)$,

(Z6)$^{t'-1}_{4.1}$ for each $k' \in [k]$, we have $W_{k'}^0 \subseteq Z_{k'}^{t'-1}$,

(Z7)$^{t'-1}_{4.1}$ for each $t'' \in [t'-1]$, we have $|\{i \in [r] : (X_i^{t'-1} \cup Y_i^{t'-1}) \cap V_{t''} \neq \emptyset\}| \leq k$.

Using that Q'_1 is a copy of K_k in R and $V(Q'_1) = \{1,\dots,k\}$, it is easy to see that (Z1)$^0_{4.1}$–(Z7)$^0_{4.1}$ hold with the above definition of X_i^0, Y_i^0, Z_i^0. We now distribute the vertices of $H_{t'}$ by setting

$$X_i^{t'} := \begin{cases} X_i^{t'-1} \cup \left(W^{t'}_{\pi_{t'}(k')} \setminus N_H^{(b_{t'}-2)k+k'}(V_0)\right) & \text{if } t' \in [t^*] \text{ and } i = p_{t'}(k') \text{ for some } k' \in [k], \\ X_i^{t'-1} & \text{otherwise,} \end{cases}$$

$$Y_i^{t'} := \begin{cases} Y_i^{t'-1} \cup \left(W^{t'}_{\pi_{t'}(k')} \setminus N_H^{(b_{t'}-2)k+k'}(V_0)\right) & \text{if } t' \in [t] \setminus [t^*] \text{ and } i = p_{t'}(k') \text{ for some } k' \in [k], \\ Y_i^{t'-1} & \text{otherwise,} \end{cases}$$

$$Z_i^{t'} := Z_i^{t'-1} \cup \bigcup_{\substack{(b,k')\in[b_{t'}-1]\times[k]:\\ i=j(t',(b-1)k+k')}} \left(W^{t'}_{\pi_{t'}(k')} \cap \left(N_H^{(b-1)k+k'}(V_0) \setminus N_H^{(b-2)k+k'}(V_0)\right)\right).$$

Let $H' := H \setminus W_0$. Recall that $N_H^{3k^3+1}(V_0)$ does not contain any vertex in W_0 (see (4.3)). Hence $N_H^i(V_0) = N_{H'}^i(V_0)$ for any $i \leq 3k^3+1$.

Note that the above definition of $X_i^{t'}, Y_i^{t'}, Z_i^{t'}$ uniquely distributes all vertices of $V^{t'}$. Indeed, first note that either $Y_i^{t'} = Y_i^{t'-1}$ for all $i \in [r]$ or $X_i^{t'} = X_i^{t'-1}$ for all $i \in [r]$ depending on whether $H_{t'} \in \mathcal{H}_c$ for some $c \in [r']$ (in which case $t' \in [t^*]$) or $H_{t'} \in \mathcal{H}_s^{\#}$ for some $s \in [q]$ or $H_{t'} \in \mathcal{H}'_{s,k'}$ for some $(s,k') \in [q] \times [k]$ (in the latter two cases we have $t' \in [t] \setminus [t^*]$). Now, consider $W_{k''}^{t'} \cap (N_H^a(V_0) \setminus N_H^{a-1}(V_0))$ for $k'' \in [k]$ and $a \in \mathbb{N}$. Note $k'' = \pi_{t'}(k')$ for some $k' \in [k]$. Then either $a > (b_{t'}-2)k + k'$ or $a \in [(b'-1)k+k'] \setminus [(b'-2)k+k']$ for some unique $b' \in [b_{t'}-1]$. Thus indeed every vertex of $V^{t'}$ belongs to exactly one of $X_i^{t'}$ or $Y_i^{t'}$ or $Z_i^{t'}$.

It is easy to see that the above definition with (4.21), (Z1)$_{4.1}^{t'-1}$ and (Z2)$_{4.1}^{t'-1}$ implies (Z1)$_{4.1}^{t'}$ and (Z2)$_{4.1}^{t'}$. Also, (Z7)$_{4.1}^{t'}$ is obvious from the construction. Moreover, (Z3)$_{4.1}^{t'-1}$ and (H3)$_{4.1}$ imply (Z3)$_{4.1}^{t'}$ while (Z6)$_{4.1}^{t'-1}$ implies (Z6)$_{4.1}^{t'}$. Similarly, we have $e_H(Y_i^{t'}, Y_j^{t'}) = 0$ if $ij \notin E(R)$. We now verify the remaining assertions of (Z4)$_{4.1}^{t'}$. First suppose that

$$E_H(X_i^{t'}, Z_{i'}^{t'}) \setminus E_H(X_i^{t'-1}, Z_{i'}^{t'-1}) \neq \emptyset \text{ or } E_H(Y_i^{t'}, Z_{i'}^{t'}) \setminus E_H(Y_i^{t'-1}, Z_{i'}^{t'-1}) \neq \emptyset.$$

Then by (H3)$_{4.1}$, we have $i = p_{t'}(k')$ for some $k' \in [k]$ and $i' = j(t', (b-1)k + k'')$ for some $k'' \in [k]$ and $b \in [b_{t'}-1]$, and H contains an edge between

$$W_{\pi_{t'}(k')}^{t'} \setminus N_H^{(b_{t'}-2)k+k'}(V_0) \text{ and } W_{\pi_{t'}(k'')}^{t'} \cap N_H^{(b-1)k+k''}(V_0).$$

This means that $(b_{t'}-2)k + k' \leq (b-1)k + k''$. Thus $b = b_{t'}-1$ and $k' \leq k''$. Moreover, since $W_{\pi_{t'}(k')}^{t'}$ is an independent set of H, we have $k' \neq k''$. Since (4.20) implies that $i = p_{t'}(k') = j(t', (b_{t'}-1)k + k')$ and $i' = j(t', (b_{t'}-2)k + k'')$ with $0 < (b_{t'}-1)k + k' - ((b_{t'}-2)k + k'') < k$, again this with (4.20) implies that $ii' \in E(R)$. Now suppose that

$$xy \in E_H(Z_i^{t'}, Z_{i'}^{t'}) \setminus E_H(Z_i^{t'-1}, Z_{i'}^{t'-1}) \text{ with } x, y \notin V_0.$$

Then by (H3)$_{4.1}$, we have $i = j(t', (b-1)k+k')$ and $i' = j(t', (b'-1)k+k'')$ for some $b, b' \in [b_t - 1]$ and $k' \neq k'' \in [k]$. However, the definition of $Z_i^{t'}$ implies that such an edge only exists when $|((b-1)k+k') - ((b'-1)k+k'')| \leq k-1$. In this case, (4.20) implies that $ii' \in E(R)$. Finally, suppose that

$$xy \in E_H(Z_i^{t'}, Z_{i'}^{t'}) \setminus E_H(Z_i^{t'-1}, Z_{i'}^{t'-1}) \text{ with } x \in V_0 \cap Z_i^{t'}.$$

Then the definition of $Z_i^{t'}$ implies that $i \in [k]$, $x \in W_i^0$ and $i' = j(t', k')$ for some $k' \in [k]$. (4.20) implies that $j(t', k') = \pi_{t'}(k')$. As $W_{\pi_{t'}(k')}^0 \cup W_{\pi_{t'}(k')}^{t'}$ is an independent set of H, we have $i \neq \pi_{t'}(k')$. However, as $R[[k]] = R[V(Q'_1)] \simeq K_k$, we know that $ii' \in E(R)$. Thus (Z4)$_{4.1}^{t'}$ holds. By the definition of $X_i^{t'}$ and $Z_i^{t'}$ with (4.21), it is obvious that (Z5)$_{4.1}^{t'}$ holds too.

Thus, by repeating this, we obtain a partition $X_1^t, \ldots, X_r^t, Y_1^t, \ldots, Y_r^t, Z_1^t, \ldots, Z_r^t$ of $V(H) \setminus W_0$ satisfying (Z1)$_{4.1}^t$–(Z7)$_{4.1}^t$. For each $i \in [r]$, let

$$X_i := X_i^t, \;\; X := X^t, \;\; Y_i := Y_i^t \setminus A, \;\; Y := Y^t \setminus A, \;\; Z'_i := Z_i^t \text{ and } Z' := Z^t.$$

Note that $A \subseteq Y^t$ by (4.9) and (Z2)$_{4.1}^t$. Moreover, X, Y, Z', A forms a partition of $V(H) \setminus W_0$. Now we consider the vertices in W_0. For each $w \in W_0$, let

$$I_w := \{i \in [r] : N_H(w) \cap (X_i \cup Y_i) \neq \emptyset\}.$$

By (4.3), we have $W_0 \cap V_0 = \emptyset$. Hence, for each vertex $w \in W_0$, there exists $t' \in [t]$ such that $w \in \widetilde{V}_{t'}$. As W_0 is an independent set, (4.3) with (H3)$_{4.1}$ implies $N_H(w) \subseteq V_{t'}$. This with (Z7)$_{4.1}^t$ implies that $|I_w| \leq k$. As $|N_R(I_w)| > 0$ by (A1)$_{4.1}$, we can assign w to Z'_i for some $i \in N_R(I_w)$. Let $Z_1, \ldots, Z_r, Z$ be the sets obtained from $Z'_1, \ldots, Z'_r, Z'$ by assigning all vertices in W_0 in this way. By (4.3), (4.9) and (Z5)$_{4.1}^t$ for each $w \in W_0$ we have $N_H(w) \subseteq X \cup Y$. Thus

for all $i \in [r]$, $w \in W_0 \cap Z_i$ *and* $x \in N_H(w)$, *we have* $x \in X_j \cup Y_j$ *for some* $j \in N_R(i)$. (4.22)

The sets X, Y, Z, A now form a partition of $V(H)$.

Step 7. Checking the properties of the partition. We now verify that this partition satisfies (B1)$_{4.1}$-(B7)$_{4.1}$. Note that (4.8) implies (B1)$_{4.1}$. Consider any $\ell \in [n']$, and let $t' \in [t]\setminus[t_*]$ and $(s,k') \in [q] \times [k]$ be such that $a_\ell \in H_{t'} \in \mathcal{H}'_{s,k'}$. Then

$$\begin{aligned} N_H(a_\ell) &\overset{(4.10)}{\subseteq} \bigcup_{k''\in[k]\setminus\{k(t')\}} W^{t'}_{k''} \setminus N_H^{3k^3+1}(V_0\cup W_0) \overset{(4.11)}{=} \bigcup_{k''\in[k]\setminus\{k'\}} W^{t'}_{\pi_{t'}(k'')} \setminus N_H^{3k^3+1}(V_0\cup W_0) \\ &\overset{(\mathrm{Z2})^t_{4.1},\,(\mathrm{Z5})^t_{4.1}}{\subseteq} \bigcup_{k''\in[k]\setminus\{k'\}} Y_{p_{t'}(k'')} \setminus N^1_H(Z) \overset{(4.1),(4.19)}{=} \bigcup_{i\in V(Q'_{s,k'})} Y_i \setminus N^1_H(Z) \overset{(4.7)}{=} \bigcup_{i\in C_\ell} Y_i\setminus N^1_H(Z). \end{aligned}$$

This proves (B2)$_{4.1}$. Moreover, whenever ℓ, t' and (s,k') are as in the proof of (B2)$_{4.1}$, for each $j' \in C_\ell$, we have $j' = p_{t'}(k'')$ for some $k'' \in [k]\setminus\{k'\}$. Thus by (4.10) and (Z2)$^t_{4.1}$, we have

$$\mathbb{E}[|N_H(a_\ell)\cap Y_{j'}|] \le \mathbb{E}[|N_H(a_\ell)\cap W^{t'}_{\pi_{t'}(k'')}|] \overset{(4.12)}{\le} \frac{2(1+1/h)m}{(k-1)n}.$$

This proves (B7)$_{4.1}$.

Properties (Z3)$^t_{4.1}$, (Z4)$^t_{4.1}$, (Z5)$^t_{4.1}$ and (4.22) imply (B3)$_{4.1}$.

For each $ij \in E(Q)$, let $s\in[r']$ and $k',k''\in[k]$ be such that $i = q_s(k')$ and $j = q_s(k'')$. Thus

$$e_H(X_i,X_j) \overset{(\mathrm{H3})_{4.1},(\mathrm{Z1})^t_{4.1}}{=} \sum_{t'\in[t^*]:H_{t'}\in\mathcal{H}_s} |E_H(W^{t'}_{\pi_{t'}(k')}, W^{t'}_{\pi_{t'}(k'')})| \pm \Delta|N_H^{3k^3}(V_0)| \overset{(\mathrm{H2})_{4.1},(\mathrm{H7})_{4.1}}{=} \frac{2m\pm\varepsilon^{1/5}n}{(k-1)r}.$$

Thus (B4)$_{4.1}$ holds. Moreover, given $i\in[r]$, let $s\in[r']$ and $k'\in[k]$ be such that $i=q_s(k')$. Then

$$|X_i| \overset{(\mathrm{Z1})^t_{4.1}}{=} \sum_{t'\in[t^*]:H_{t'}\in\mathcal{H}_s} |W^{t'}_{\pi_{t'}(k')}| \pm |N_H^{3k^3}(V_0)| \overset{(\mathrm{H7})_{4.1}}{=} \tilde{n} - \varepsilon^{1/3}n/r \pm \eta^{1/3}n.$$

Similarly, for $i\in[r]$, since by (4.9) the vertices of A only belong to $V(H_{t'})$ for $t'\in[t]\setminus[t_*]$,

$$\begin{aligned} |Y_i| &\overset{(\mathrm{Z2})^t_{4.1}}{=} \sum_{(t',k'):p_{t'}(k')=i,t'\in[t]\setminus[t^*]} |W^{t'}_{\pi_{t'}(k')}\setminus A| \pm |N_H^{3k^3}(V_0)| \\ &\overset{(4.19)}{=} \sum_{(s,k'):q'_s(k')=i}\ \sum_{t':H_{t'}\in\mathcal{H}^\#_s} |W^{t'}_{\pi_{t'}(k')}| + \sum_{(s,k'):q'_s(k')=i}\ \sum_{k''\in[k]}\ \sum_{t':H_{t'}\in\mathcal{H}'_{s,k''}} |W^{t'}_{\pi_{t'}(k')}\setminus A| \pm \eta^{1/2}n \\ &\overset{(\mathrm{H8})_{4.1},(4.13)}{=} \sum_{(s,k'):q'_s(k')=i} \mathrm{mult}_{\mathcal{F}^\#}(V(Q'_s)) + |\tilde{Y}_i| \pm 2q\eta^{2/5}n = d_{\mathcal{F}^\#}(i) + |\tilde{Y}_i| \pm 2q\eta^{2/5}n \\ &\overset{(4.15),(4.16)}{=} n_i - \tilde{n} + \varepsilon^{1/3}n/r \pm \eta^{1/3}n. \end{aligned}$$

Together with (4.3), (Z5)$^t_{4.1}$ and (H2)$_{4.1}$, this now implies that for each $i\in[r]$

$$|X_i| + |Y_i| + |Z_i| = n_i \pm \eta^{1/4}n.$$

Also, the definition of $\tilde{n}$ with (A4)$_{4.1}$ implies that $|Y_i| \le 2\varepsilon^{1/3}n/r$. Thus (B5)$_{4.1}$ holds. Finally, (4.3) and (Z5)$_{4.1}$ imply (B6)$_{4.1}$. □

5. Packing graphs into a super-regular blow-up

In this section, we prove our main lemma. Roughly speaking, this lemma says the following. Suppose we have disjoint vertex sets V, Res_t and V_0 and suppose that we have a super-regular K_k-factor blow-up $G[V]$ on vertex set V, and suitable graphs $G[Res_t]$, $G[V, Res_t]$, $F[V, Res_t]$ and $F'[Res_t, V_0]$ are also provided. Then we can pack an appropriate collection $\mathcal{H}$ of graphs into $G\cup F\cup F'$. Here V_0 is the exceptional set obtained from an application of Szemerédi's regularity lemma and Res_t is a suitable 'reservoir' set where V_0 is much smaller than Res_t, which in turn is much smaller than V. The k-cliques provided by the multi-k-graph $\mathcal{C}^*_t$ below will allow us to find a suitable embedding of the neighbours of the vertices mapped to V_0. When we apply Lemma 5.1 in Section 6, the reservoir set Res_t will play the role of the set $U\cup U_0$ below. U_0 will correspond to a set of exceptional vertices in Res_t. (A9)$_{5.1}$ will allow us to embed the neighbours of the vertices mapped to U_0.

Note that the packing ϕ is designed to cover most of the edges of the blown-up K_k-factor $G[V]$, but only covers a small proportion of the edges of G incident to U. (A7)$_{5.1}$ provides the edges incident to the vertices mapped to V_0, and (A8)$_{5.1}$ allows us to embed the neighbourhoods of these vertices.

Lemma 5.1. *Suppose $n, n', k, \Delta, r, T \in \mathbb{N}$ with $0 < 1/n, 1/n' \ll \eta \ll \varepsilon \ll 1/T \ll \alpha \ll d \ll 1/k, \sigma, \nu, 1/\Delta < 1$ and $\eta \ll 1/r \ll \sigma$ and $k \mid r$. Suppose that R and Q are graphs with $V(R) = V(Q) = [r]$ such that Q is a union of r/k vertex-disjoint copies of K_k. Suppose that $V_0, \dots, V_r, U_0, \dots, U_r$ is a partition of a set of n vertices such that $|V_0| \leq \varepsilon n$, $|U_0| \leq \varepsilon n$ and for all $i \in [r]$*

$$n' = |V_i| = \frac{(1 - 1/T \pm 2\varepsilon)n}{r} \quad \text{and} \quad |U_i| = \frac{(1 \pm 2\varepsilon)n}{Tr}.$$

Let $V := \bigcup_{i\in[r]} V_i$ and $U := \bigcup_{i\in[r]} U_i$. Suppose that G, F, F' are edge-disjoint graphs such that $V(G) = V \cup U \cup U_0$, F is a bipartite graph with vertex partition (V, U), and F' is a bipartite graph with vertex partition (V_0, U) such that $F' = \bigcup_{t\in[T]} \bigcup_{v\in V_0} F'_{v,t}$, where all the $F'_{v,t}$ are pairwise edge-disjoint stars with centre v.

Suppose that $\mathcal{H}$ is a collection of (k, η)-chromatic η-separable graphs on n vertices, and for each $t \in [T]$ we have a multi-$(k-1)$-graph $\mathcal{C}_t$ on $[r]$ and a multi-k-graph $\mathcal{C}^*_t$ on $[r]$ with $E(\mathcal{C}_t) = \{C_{v,t} : v \in V_0\}$ and $E(\mathcal{C}^*_t) = \{C^*_{v,t} : v \in V_0\}$. Assume the following hold.

(A1)$_{5.1}$ *For each $H \in \mathcal{H}$, we have $\Delta(H) \leq \Delta$ and $e(H) \geq n/4$,*
(A2)$_{5.1}$ *$n^{7/4} \leq e(\mathcal{H}) \leq (1 - \nu)(k - 1)\alpha n^2/(2r)$,*
(A3)$_{5.1}$ *$G[V]$ is $(T^{-1/2}, \alpha)$-super-regular with respect to the vertex partition $(Q, V_1, \dots, V_r)$,*
(A4)$_{5.1}$ *for each $ij \in E(R)$, the graphs $G[V_i, U_j]$ and $G[U_i, U_j]$ are both $(\varepsilon^{1/50}, (d^3))^+$-regular,*
(A5)$_{5.1}$ *$\delta(R) \geq (1 - 1/k + \sigma)r$,*
(A6)$_{5.1}$ *for all $ij \in E(Q)$ and $u \in U_i$, we have $d_{F,V_j}(u) \geq d^3 n'$,*
(A7)$_{5.1}$ *for all $v \in V_0$ and $t \in [T]$ and $i \in C_{v,t}$, we have $d_{F'_{v,t},U_i}(v) \geq (1 - d)\alpha|U_i|$,*
(A8)$_{5.1}$ *for all $v \in V_0$ and $t \in [T]$, we have $C_{v,t} \subseteq C^*_{v,t}$, $R[C^*_{v,t}] \simeq K_k$, and $\Delta(\mathcal{C}^*_t) \leq \frac{\varepsilon^{3/4}n}{r}$,*
(A9)$_{5.1}$ *for each $u \in U_0$, we have*

$$|\{i \in [r] : d_{G,V_j}(u) \geq d^3 n' \text{ for all } j \in N_Q(i)\}| > \varepsilon^{1/4} r.$$

Then there exists a packing ϕ of $\mathcal{H}$ into $G \cup F \cup F'$ such that

(B1)$_{5.1}$ *$\Delta(\phi(\mathcal{H})) \leq 4k\Delta\alpha n/r$,*
(B2)$_{5.1}$ *for each $u \in U$, we have $d_{\phi(\mathcal{H})\cap G}(u) \leq 2\Delta\varepsilon^{1/8} n/r$,*
(B3)$_{5.1}$ *for each $i \in [r]$, we have $e_{\phi(\mathcal{H})\cap G}(V_i, U \cup U_0) < \varepsilon^{1/2} n^2/r^2$.*

Roughly, the proof of Lemma 5.1 will proceed as follows. In Step 1 we define a partition of U_0 and an auxiliary digraph D. In Step 2 we define a partition of each $H \in \mathcal{H}$. For each graph $H \in \mathcal{H}$ we apply Lemma 4.1 to partition $V(H)$ into X^H, Y^H, Z^H, A^H. We will embed A^H into V_0 and the remainder of H into $V \cup U \cup U_0$. In Step 3, we apply Lemma 3.6 to find an appropriate function ϕ' packing $\{H[Y^H \cup Z^H \cup A^H] : H \in \mathcal{H}\}$ into $G[U] \cup F'$. Guided by the auxiliary digraph D, in Step 4 we modify the partition by removing a suitable W^H from X^H (so that we can later embed $X^H \backslash W^H$ into V). We will also find a function ϕ'' packing $\{H[W^H] : H \in \mathcal{H}\}$ into $G[U]$ in an appropriate way, which ensures that later we can also pack $\{H[X^H \setminus W^H, W^H] : H \in \mathcal{H}\}$ into $F[V, U] \cup G[V, U]$. In Step 5 we will partition $\mathcal{H}$ into subcollections $\mathcal{H}_{1,1}, \dots, \mathcal{H}_{T,w}$ and use Lemma 3.14 to pack $\{H[X^H \setminus W^H] : H \in \mathcal{H}_{t,w'}\}$ into an internally q-regular graph $H_{t,w'}$ (for some suitable q). Finally, in Step 6 we apply the blow-up lemma for approximate decompositions (Theorem 3.15) to pack $\{H_{t,w'} : t \in [T], w' \in [w]\}$ into $G[V]$ such that the packing obtained is consistent with $\phi' \cup \phi''$.

Proof. Let $r' := r/k$ and $Q_1, \dots, Q_{r'}$ be the copies of K_k in Q. Let $n_0 := |V_0|$ and $V_0 =: \{v_1, \dots, v_{n_0}\}$. By (A1)$_{5.1}$, for each $H \in \mathcal{H}$, we have

$$e(H) \leq \Delta n. \tag{5.1}$$

Moreover,

$$\kappa := |\mathcal{H}| \overset{(A1)_{5.1},(A2)_{5.1}}{\leq} 2(1-\nu)(k-1)\alpha n/r. \tag{5.2}$$

Step 1. Partition of U_0 and the construction of an auxiliary digraph D. In Step 2, we will find a partition of each $H \in \mathcal{H}$ which closely reflects the structure of G. However we need the partitions to match up exactly. The following auxiliary graph will enable us to carry out this adjustment in Step 4. Let D be the directed graph with $V(D) = [r]$ and

$$E(D) = \{\vec{ij} : i \neq j \in [r], N_Q(i) \subseteq N_R(j)\}. \tag{5.3}$$

For each $ij \in E(R)$, we let

$$U_i(j) := \{u \in U_i : d_{G,V_j}(u) \geq (d^3 - \varepsilon^{1/50})n'\}.$$

Then $(A4)_{5.1}$ with Proposition 3.4 implies that $|U_i(j)| \geq (1 - 2\varepsilon^{1/50})|U_i|$. For each $\vec{ij} \in E(D)$, we define

$$U_j^D(i) := \bigcap_{i' \in N_Q(i)} U_j(i'), \tag{5.4}$$

then we have

$$|U_j^D(i)| \geq (1 - 2(k-1)\varepsilon^{1/50})|U_j| \geq n/(2Tr). \tag{5.5}$$

In Step 4 we will map some vertices $x \in V(H)$ whose 'natural' image would have been in V_i to $U_j^D(i)$ instead, in order to 'balance out' the vertex class sizes.

Claim 3. *There exists a set $I^* = \{i_1^*, \dots, i_k^*\} \subseteq [r]$ of k distinct numbers such that for any $k' \in [k]$ and $j \in [r]$, there exists a directed path $P(i_{k'}^*, j)$ from $i_{k'}^*$ to j in D.*

Proof. First, we claim that all $i \neq j \in [r]$ satisfy that $N_D^-(i) \cap N_D^-(j) \neq \emptyset$. Indeed, as $|N_R(\{i,j\})| \geq 2\delta(R) - r \geq (1 - 2/k + 2\sigma)r$, we have that

$$|\{s \in [r'] : |N_{R,V(Q_s)}(\{i,j\})| \geq k-1\}| \geq \sigma r \geq 3.$$

Thus there exists $s \in [r']$ such that $i, j \notin V(Q_s)$ while $|N_{R,V(Q_s)}(\{i,j\})| \geq k-1$. We choose $j' \in V(Q_s)$ such that $Q_s \setminus \{j'\} \subseteq N_R(\{i,j\})$, then (5.3) implies that $i, j \in N_D^+(j')$.

Now, we consider a number $i \in [r]$ which maximizes $|A(i)|$, where

$$A(i) = \{j \in [r] : \text{ there exists a directed path from } i \text{ to } j \text{ in } D\}.$$

If there exists $j \in [r]$ such that $j \notin A(i)$, then by the above claim, there exists $j' \in [r]$ such that $i, j \in N_D^+(j')$. Then $A(i) \cup \{j\} \subseteq A(j')$, which is a contradiction to the maximality of $A(i)$. Thus, we have $A(i) = [r]$. Let $i_1^* := i$.

Since $d_R(i_1^*) \geq \delta(R) \geq (1 - 1/k + \sigma)r$ by $(A5)_{5.1}$, we have $|\{s \in [r'] : N_{R,V(Q_s)}(i_1^*) = k\}| \geq \sigma r$. Thus, there exists $s \in [r']$ such that $V(Q_s) \subseteq N_R(i_1^*)$, and this with (5.3) implies that $V(Q_s) \subseteq N_D^-(i_1^*)$. We let $i_2^*, \dots, i_k^*$ be $k-1$ arbitrary numbers in $V(Q_s)$. Then for all $k' \in [k]$ and $j \in [r]$, there exists a directed path from $i_{k'}^*$ to i_1^* and a directed path from i_1^* to j in D. Thus there exists a directed path from $i_{k'}^*$ to j in D. This proves the claim. □

We will now determine the approximate class sizes $\tilde{n}_i$ that our partition of H will have. For this, we first partition U_0 into $U_1', \dots, U_r'$ in such a way that the vertices in U_i' are 'well connected' to the blow-up of the k-clique in Q to which i belongs.

$$\text{For all } i \in [r], u \in U_i' \text{ and } j \in N_Q(i), \text{ we have } d_{G,V_j}(u) \geq d^3 n' \text{ and } |U_i'| \leq 2\varepsilon^{3/4} n/r. \tag{5.6}$$

Indeed, it is easy to greedily construct such a partition by using the fact that $|U_0| \leq \varepsilon n$ and $(A9)_{5.1}$.

For $i \in I^*$, we will slightly increase the partition class sizes (cf. (5.9) and $(X5)_{5.1}$) as this will allow us to subsequently move any excess vertices from classes corresponding to I^* to another arbitrary class via the paths provided by Claim 3. For each $i \in [r]$, we let

$$n_i := n' + |U_i| + |U_i'| = |V_i| + |U_i| + |U_i'|, \tag{5.7}$$

then we have

$$n_i = (1 - 1/T \pm 2\varepsilon)n/r + (1 \pm 2\varepsilon)n/(Tr) \pm 2\varepsilon^{3/4}n/r = (1 \pm \varepsilon^{2/3}/2)n/r \text{ and } \sum_{i\in[r]} n_i = n - n_0. \tag{5.8}$$

For each $i \in [r]$ we let

$$\tilde{n}_i := \begin{cases} n_i + (r'-1)\eta^{1/5}n & \text{if } i \in I^*, \\ n_i - \eta^{1/5}n & \text{if } i \in [r] \setminus I^*. \end{cases} \tag{5.9}$$

This with (5.8) implies that for each $i \in [r]$,

$$\tilde{n}_i = \frac{(1 \pm \varepsilon^{2/3})n}{r} \text{ and } \sum_{i\in[r]} \tilde{n}_i = \sum_{i\in[r]} n_i = n - n_0. \tag{5.10}$$

Step 2. Preparation of the graphs in $\mathcal{H}$. First, we will partition $\mathcal{H}$ into T collections $\mathcal{H}_1, \dots, \mathcal{H}_T$. Later we will pack each $\mathcal{H}_t$ into $G \cup F \cup \bigcup_{v\in V_0} F'_{v,t}$. (Recall that the $F'_{v,t}$ form a decomposition of F'.) As $G \cup F \cup F'$ has vertex partition $V_0, \dots, V_r, U_1, \dots, U_r, U'_1, \dots, U'_r$, for each $H \in \mathcal{H}$, we also need a suitable partition of $V(H)$ which is compatible with the partition of the host graph $G \cup F \cup F'$. To achieve this, we will apply Lemma 4.1 to each graph $H \in \mathcal{H}_t$ with the hypergraphs $\mathcal{C}_t$ and $\mathcal{C}^*_t$ to find the desired partition of $V(H)$.

By (5.1) we can partition $\mathcal{H}$ into $\mathcal{H}_1, \dots, \mathcal{H}_T$ such that for each $t \in [T]$,

$$\begin{aligned} e(\mathcal{H}_t) &= e(\mathcal{H})/T \pm \Delta n \overset{(\mathrm{A2})_{5.1}}{\leq} (1 - 2\nu/3)\alpha(k-1)n^2/(2Tr), \text{ and} \\ |\mathcal{H}_t| &\overset{(\mathrm{A1})_{5.1}}{\leq} 4e(\mathcal{H}_t)/n \leq 2\alpha(k-1)n/(Tr). \end{aligned} \tag{5.11}$$

For each $t \in [T]$, we wish to apply the randomised algorithm given by Lemma 4.1 with the following objects and parameters independently for all $H \in \mathcal{H}_t$.

object/parameter	H	R	Q	$\mathcal{C}_t$	$\mathcal{C}^*_t$	n_0	$C_{v_\ell,t}$	$C^*_{v_\ell,t}$	$\lceil 3/d \rceil$	η	ε	k	Δ	r	$\tilde{n}_i$
playing the role of	H	R	Q	$\mathcal{F}$	$\mathcal{F}^*$	n'	C_ℓ	C^*_ℓ	h	η	ε	k	Δ	r	n_i

Indeed, $(\mathrm{A5})_{5.1}$, $(\mathrm{A8})_{5.1}$ imply that $(\mathrm{A1})_{4.1}$, $(\mathrm{A2})_{4.1}$ and $(\mathrm{A3})_{4.1}$ hold with the above objects and parameters, respectively. Moreover, (5.10) implies that $(\mathrm{A4})_{4.1}$ holds too. Thus we obtain a partition $X^H_1, \dots, X^H_r, Y^H_1, \dots, Y^H_r, Z^H_r, \dots, Z^H_r, A^H$ of $V(H)$ such that $A^H = \{a^H_1, \dots, a^H_{n_0}\}$ is a 3-independent set of H and the following hold, where $X^H := \bigcup_{i\in[r]} X^H_i$, $Y^H := \bigcup_{i\in[r]} Y^H_i$, and $Z^H := \bigcup_{i\in[r]} Z^H_i$.

$(\mathrm{X1})_{5.1}$ For each $\ell \in [n_0]$, we have $d_H(a^H_\ell) \leq \frac{(2+d)e(H)}{n}$,

$(\mathrm{X2})_{5.1}$ for each $\ell \in [n_0]$, we have $N_H(a^H_\ell) \subseteq \bigcup_{i\in C_{v_\ell,t}} Y^H_i \setminus N^1_H(Z^H)$,

$(\mathrm{X3})_{5.1}$ $H[X^H]$ admits the vertex partition $(Q, X^H_1, \dots, X^H_r)$, and $H \setminus E(H[X^H])$ admits the vertex partition $(R, X^H_1 \cup Y^H_1 \cup Z^H_1, \dots, X^H_r \cup Y^H_r \cup Z^H_r)$,

$(\mathrm{X4})_{5.1}$ for each $ij \in E(Q)$, we have $e_H(X^H_i, X^H_j) = \frac{2e(H) \pm \varepsilon^{1/5}n}{(k-1)r}$,

$(\mathrm{X5})_{5.1}$ for each $i \in [r]$, we have $|Y^H_i| \leq 2\varepsilon^{1/3}n/r$ and $|X^H_i| + |Y^H_i| + |Z^H_i| = \tilde{n}_i \pm \eta^{1/4}n$; in particular, this with (5.9) implies that for each $i \in [r]$, we have

$$\widehat{n}^H_i := |X^H_i| + |Y^H_i| + |Z^H_i| \in \begin{cases} [n_i, n_i + \eta^{1/6}n] & \text{if } i \in I^*, \\ [n_i - \eta^{1/6}n, n_i] & \text{otherwise,} \end{cases}$$

$(\mathrm{X6})_{5.1}$ $N^1_H(X^H) \setminus X^H \subseteq Z^H$, and $|Z^H| \leq 4\Delta^{3k^3}\eta^{0.9}n$,

$(\mathrm{X7})_{5.1}$ for all $\ell \in [n_0]$ and $i \in C_{v_\ell,t}$, we have $\mathbb{E}[N_H(a^H_\ell) \cap Y^H_i] \leq \frac{(2+d)e(H)}{(k-1)n}$.

By applying this randomised algorithm independently for each $H \in \mathcal{H}_1 \cup \dots \cup \mathcal{H}_T$, we obtain that for all $t \in [T]$, $\ell \in [n_0]$ and $i \in C_{v_\ell,t}$, we have $\mathbb{E}[\sum_{H\in\mathcal{H}_t} |N_H(a^H_\ell) \cap Y^H_i|] \leq \frac{(2+d)e(\mathcal{H}_t)}{(k-1)n}$. Note that for each $H \in \mathcal{H}_t$, we have $|N_H(a^H_\ell) \cap Y^H_i| \leq \Delta$. As our applications of the randomised algorithm are independent for all $H \in \mathcal{H}_t$, a Chernoff bound (Lemma 3.1) together with (A2)$_{5.1}$ implies that for all $t \in [T]$, $\ell \in [n_0]$ and $i \in C_{v_\ell,t}$, we have

$$\mathbb{P}\Big[\sum_{H\in\mathcal{H}_t} |N_H(a_\ell^H)\cap Y_i^H| \geq \frac{2(1+d)e(\mathcal{H}_t)}{(k-1)n}\Big] \leq 2\exp\Big(-\frac{d^2e(\mathcal{H}_t)^2/((k-1)^2n^2)}{2\Delta^2|\mathcal{H}_t|}\Big) \overset{(5.11),(\mathrm{A2})_{5.1}}{\leq} e^{-n^{1/3}}.$$

By taking a union bound over all $t \in [T], \ell \in [n_0]$ and $i \in C_{v_\ell,t}$, we can show that the following property (X8)$_{5.1}$ holds with probability at least $1 - kTn_0e^{-n^{1/3}} > 0$.

(X8)$_{5.1}$ For all $t \in [T]$, $\ell \in [n_0]$ and $i \in C_{v_\ell,t}$, we have $\sum_{H\in\mathcal{H}_t} |N_H(a_\ell^H)\cap Y_i^H| \leq \frac{2(1+d)e(\mathcal{H}_t)}{(k-1)n}$.

Thus we conclude that for all $H \in \mathcal{H}$ there exist partitions $X_1^H,\dots,X_r^H, Y_1^H,\dots,Y_r^H, Z_r^H,\dots,Z_r^H$, A^H of $V(H)$ such that $A^H = \{a_1^H,\dots,a_{n_0}^H\}$ is a 3-independent set of H and such that (X1)$_{5.1}$–(X6)$_{5.1}$ and (X8)$_{5.1}$ hold.

Note that $\sum_{i\in[r]} \widehat{n}_i^H = |V(H)| - |A^H| = n - n_0$. This with (5.8) implies that for each $H \in \mathcal{H}$, we have

$$\sum_{i\in I^*}(\widehat{n}_i^H - n_i) = \sum_{i\in[r]\setminus I^*}(n_i - \widehat{n}_i^H). \tag{5.12}$$

The following claim determines the number of vertices that we will redistribute via D.

Claim 4. *For each $H \in \mathcal{H}$, there exists a function $f^H : E(D) \to [\eta^{1/7}n] \cup \{0\}$ such that for each $i \in [r]$, we have*

$$\sum_{j\in N_D^+(i)} f^H(\vec{ij}) - \sum_{j\in N_D^-(i)} f^H(\vec{ji}) = \widehat{n}_i^H - n_i.$$

Proof. By (X5)$_{5.1}$, for each $i \in I^*$, we have $\widehat{n}_i^H - n_i \geq 0$ and for each $i \in [r] \setminus I^*$, we have $n_i - \widehat{n}_i^H \geq 0$. Thus by (5.12), there exists a bijection g^H from

$$\bigcup_{i\in I^*}\{i\}\times[\widehat{n}_i^H - n_i] \text{ to } \bigcup_{i\in[r]\setminus I^*}\{i\}\times[n_i - \widehat{n}_i^H].$$

For all $i \in I^*$ and $m \in [\widehat{n}_i^H - n_i]$, let $g^H(i,m) =: (g_1^H(i,m), g_2^H(i,m))$ and let $P_{i,m}$ be a directed path from i to $g_1^H(i,m)$ in D, which exists by Claim 3. As g^H is a bijection, for each $i \in [r]$, we have

$$|(g_1^H)^{-1}(i)| = \begin{cases} 0 & \text{if } i \in I^*,\\ n_i - \widehat{n}_i^H & \text{otherwise.}\end{cases} \tag{5.13}$$

For each $\vec{ij} \in E(D)$, we let

$$f^H(\vec{ij}) := |\{(i',m) : i' \in I^*, m \in [\widehat{n}_{i'}^H - n_{i'}] \text{ and } \vec{ij} \in E(P_{i',m})\}|.$$

Then for each $\vec{ij} \in E(D)$, we have

$$f^H(\vec{ij}) \leq \Big|\bigcup_{i'\in I^*}\{i'\}\times[\widehat{n}_{i'}^H - n_{i'}]\Big| \overset{(\mathrm{X5})_{5.1}}{\leq} k\eta^{1/6}n \leq \eta^{1/7}n.$$

Note that for any $i \in I^*$ and $m \in [\widehat{n}_i^H - n_i]$, the path $P_{i,m}$ starts from a vertex in I^* and ends at $[r] \setminus I^*$. Thus for each $i \in [r]$ we have

$$\begin{aligned}&\sum_{j\in N_D^+(i)} f^H(\vec{ij}) - \sum_{j\in N_D^-(i)} f^H(\vec{ji})\\ &= |\{(i',m) : m \in [\widehat{n}_{i'}^H - n_{i'}], i = i' \in I^*\}| - |\{(i',m) : i' \in I^*, m \in [\widehat{n}_{i'}^H - n_{i'}], g_1^H(i',m) = i\}|\\ &= \begin{cases} (\widehat{n}_i^H - n_i) - 0 = \widehat{n}_i^H - n_i & \text{if } i \in I^*,\\ 0 - (g_1^H)^{-1}(i) \overset{(5.13)}{=} \widehat{n}_i^H - n_i & \text{otherwise.}\end{cases}\end{aligned}$$

This proves the claim. □

For each $H \in \mathcal{H}$, we fix a function f^H satisfying Claim 4. For each $\vec{ij} \notin E(D)$, it will be convenient to set $f^H(\vec{ij}) := 0$.

We aim to embed vertices in $X_i^H \cup Y_i^H \cup Z_i^H$ into $V_i \cup U_i \cup U_i'$. As $|V_i \cup U_i \cup U_i'| = n_i$, by (5.7), it would be ideal if $|X_i^H \cup Y_i^H \cup Z_i^H| = n_i$ and $|X_i^H| = n'$. However, (X5)$_{5.1}$ only guarantees that this is approximately true. In order to deal with this, we will use D and f^H to assign a small number of 'excess' vertices $u \in X_i^H$ into U_j when $\vec{ij} \in E(D)$. The definition of D will ensure that the image of u still has many neighbours in $V_{i'}$ for all $i' \in N_Q(i)$.

Step 3. Packing the graphs $H[Y^H \cup Z^H \cup A^H]$ into $G[U] \cup F'$. Now, we aim to find a suitable function ϕ' which packs $\{H[Y^H \cup Z^H \cup A^H] : H \in \mathcal{H}\}$ into $G[U] \cup F'$. In order to find ϕ', we will use Lemma 3.6. Moreover, we choose ϕ' in such a way that we can later extend ϕ' into a packing of the entire graphs $H \in \mathcal{H}$. One important property we need to ensure is the following: for any vertex $x \in X_j^H$ which is not embedded by ϕ', and any vertices $y_1, \dots, y_i \in N_H(x) \cap (Y^H \cup Z^H)$ which are already embedded by ϕ', we need $N_G(\phi'(\{y_1, \dots, y_i\})) \cap V_j$ to be large, so that x can be later embedded into $N_G(\phi'(\{y_1, \dots, y_i\})) \cap V_j$. For this, we will introduce a hypergraph $\mathcal{N}_H$ which encodes information about the set $N_H(x) \cap (Y^H \cup Z^H)$ for each vertex $x \in X^H$. In order to describe the structure of G and H more succinctly, we also introduce a graph R' on $[2r]$ such that
$$E(R') = \{ij : (i-r)(j-r) \in E(R) \text{ or } i(j-r) \in E(R)\}.$$
For all $i \in [r]$ and $H \in \mathcal{H}$, let $V_{i+r} := U_i$ and $X_{i+r}^H := Y_i^H \cup Z_i^H$. Note that (X3)$_{5.1}$ and (A4)$_{5.1}$ imply that for each $H \in \mathcal{H}$,
$$\begin{aligned}&H[Y^H \cup Z^H] \textit{ admits the vertex partition } (R', \emptyset, \dots, \emptyset, X_{r+1}^H, \dots, X_{2r}^H), \textit{ and}\\ &G \textit{ is } (\varepsilon^{1/50}, (d^3))^+\textit{-regular with respect to the partition } (R', V_1, \dots, V_{2r}).\end{aligned} \tag{5.14}$$
For all $H \in \mathcal{H}$ and $x \in X^H$, let
$$e_{H,x} := N_H(x) \setminus X^H \overset{(X6)_{5.1}}{=} N_H(x) \cap Z^H.$$
Let $\mathcal{N}_H$ be a multi-hypergraph on vertex set Z^H with
$$E(\mathcal{N}_H) := \{e_{H,x} : x \in N_H^1(Z^H) \cap X^H\}, \tag{5.15}$$
and let $f_H : E(\mathcal{N}_H) \to [r]$ be a function such that for all $x \in X^H$, we have that $x \in X^H_{f_H(e_{H,x})}$. Then $\Delta(\mathcal{N}_H) \leq \Delta$ and $\mathcal{N}_H$ has edge-multiplicity at most Δ. Note that, as $\mathcal{N}_H$ is a multi-hypergraph, there could be two distinct vertices $x \neq x' \in X^H$ such that $e_{H,x}$ and $e_{H,x'}$ consists of exactly the same vertices while $f_H(e_{H,x}) \neq f_H(e_{H,x'})$.

Our next aim is to construct a function ϕ' which packs $\{H[Y^H \cup Z^H \cup A^H] : H \in \mathcal{H}\}$ into $G[U] \cup F'$ in such a way that the following hold for all $H \in \mathcal{H}$.

(Φ'1)$_{5.1}$ For each $e \in E(\mathcal{N}_H)$, we have $|N_G(\phi'(e)) \cap V_{f_H(e)}| \geq d^{5\Delta}|V_{f_H(e)}|$,
(Φ'2)$_{5.1}$ for each $v \in V(G)$, we have $|\{H \in \mathcal{H} : v \in \phi'(H[Y^H \cup Z^H])\}| \leq \varepsilon^{1/8} n/r$,
(Φ'3)$_{5.1}$ for all $i \in [r]$ and $H \in \mathcal{H}$, we have $\phi'(Y_i^H \cup Z_i^H) \subseteq U_i$, and
(Φ'4)$_{5.1}$ $\phi'(A^H) = V_0$.

Claim 5. *There exists a function ϕ' packing $\{H[Y^H \cup Z^H \cup A^H] : H \in \mathcal{H}\}$ into $G[U] \cup F'$ which satisfies* (Φ'1)$_{5.1}$–(Φ'4)$_{5.1}$.

Proof. Let $\phi'_0 : \emptyset \to \emptyset$ be an empty packing. Let $H_1, \dots, H_\kappa$ be an enumeration of $\mathcal{H}$. For each $s \in [\kappa]$, let
$$\mathcal{H}^s := \{H_{s'}[Y^{H_{s'}} \cup Z^{H_{s'}} \cup A^{H_{s'}}] : s' \in [s]\}.$$
Our aim is to successively extend ϕ'_0 into $\phi'_1, \dots, \phi'_\kappa$ in such a way that each ϕ'_s satisfies the following.

(Φ'1)$^s_{5.1}$ ϕ'_s packs $\mathcal{H}^s$ into $G[U] \cup F'$,
(Φ'2)$^s_{5.1}$ for all $s' \in [s]$ and $e \in E(\mathcal{N}_{H_{s'}})$, we have $|N_G(\phi'_s(e)) \cap V_{f_{H_{s'}}(e)}| \geq d^{5\Delta}|V_{f_{H_{s'}}(e)}|$,
(Φ'3)$^s_{5.1}$ for each $v \in V(G)$, we have $|\{s' \in [s] : v \in \phi'_s(H_{s'}[Y^{H_{s'}} \cup Z^{H_{s'}}])\}| \leq \varepsilon^{1/8} n/r$,
(Φ'4)$^s_{5.1}$ for all $i \in [2r] \setminus [r]$ and $s' \in [s]$, we have $\phi'_s(X_i^{H_{s'}}) \subseteq V_i$,

$(\Phi'5)^s_{5.1}$ for all $s' \in [s]$ and $\ell \in [n_0]$, we have $\phi'_s(a_\ell^{H_{s'}}) = v_\ell$,

$(\Phi'6)^s_{5.1}$ for all $s' \in [s]$, $t \in [T]$ with $H_{s'} \in \mathcal{H}_t$, we have $\phi'_s(H_{s'}[Y^{H_{s'}} \cup Z^{H_{s'}} \cup A^{H_{s'}}]) \subseteq G[U] \cup \bigcup_{v \in V_0} F'_{v,t}$.

Note that ϕ'_0 vacuously satisfies $(\Phi'1)^0_{5.1}$–$(\Phi'6)^0_{5.1}$. Assume we have already constructed ϕ'_s satisfying $(\Phi'1)^s_{5.1}$–$(\Phi'6)^s_{5.1}$ for some $s \in [\kappa - 1] \cup \{0\}$. We will show that we can construct ϕ'_{s+1}. Let

$$G(s) := G \setminus \phi'_s(\mathcal{H}^s).$$

For all $\ell \in [n_0]$ and $a_\ell^{H_{s+1}} \in A^{H_{s+1}}$, we first let

$$\psi(a_\ell^{H_{s+1}}) := v_\ell. \tag{5.16}$$

For each $i \in [2r] \setminus [r]$, let

$$V_i^{\mathrm{bad}} := \Big\{ v \in V_i : |\{s' \in [s] : v \in \phi'_{s'}(H_{s'}[Y^{H_{s'}} \cup Z^{H_{s'}}])\}| \geq \frac{\varepsilon^{1/8} n}{r} - 1 \Big\}.$$

Note that

$$|V_i^{\mathrm{bad}}| \overset{(\Phi'4)^s_{5.1}}{\leq} \frac{\sum_{s' \in [s]} |Y_{i-r}^{H_{s'}} \cup Z_{i-r}^{H_{s'}}|}{\frac{\varepsilon^{1/8} n}{r} - 1} \overset{(X5)_{5.1},(X6)_{5.1}}{\leq} 3\varepsilon^{1/3 - 1/8}\kappa \overset{(5.2)}{\leq} \frac{\varepsilon^{1/5} n}{r}. \tag{5.17}$$

Let $t \in [T]$ be such that $H_{s+1} \in \mathcal{H}_t$. For all $i \in [2r] \setminus [r]$ and $x \in X_i^{H_{s+1}}$, we let

$$B_x := \begin{cases} N_{F'_{v_\ell,t}, V_i}(v_\ell) \setminus (N_{\phi'_s(\mathcal{H}^s)}(v_\ell) \cup V_i^{\mathrm{bad}}) & \text{if } x \in N_{H_{s+1}}(a_\ell^{H_{s+1}}) \cap X_i^{H_{s+1}} \text{ for some } \ell \in [n_0], \\ V_i \setminus V_i^{\mathrm{bad}} & \text{otherwise.} \end{cases}$$

We will later embed x into B_x. Note that if $x \in N_{H_{s+1}}(a_\ell^{H_{s+1}})$, then $x \notin N_{H_{s+1}}(a_{\ell'}^{H_{s+1}})$ for any $\ell' \in [n_0] \setminus \{\ell\}$ as $A^{H_{s+1}}$ is a 3-independent set in H_{s+1}. Also, if $x \in N_{H_{s+1}}(a_\ell^{H_{s+1}}) \cap X_i^{H_{s+1}}$, then by $(X2)_{5.1}$ we have $i - r \in C_{v_\ell,t}$. Thus in this case

$$\begin{aligned} |B_x| &\geq d_{F'_{v_\ell,t}, V_i}(v_\ell) - d_{\phi'_s(\mathcal{H}^s) \cap F'_{v_\ell,t}, V_i}(v_\ell) - |V_i^{\mathrm{bad}}| \\ &\overset{(A7)_{5.1},(5.17)}{\geq} (1-d)\alpha|U_{i-r}| - d_{\phi'_s(\mathcal{H}^s) \cap F'_{v_\ell,t}, V_i}(v_\ell) - \varepsilon^{1/5} n/r \\ &\overset{\substack{(X2)_{5.1},(\Phi'4)^s_{5.1},\\ (\Phi'5)^s_{5.1},(\Phi'6)^s_{5.1}}}{\geq} (1-d)\alpha|U_{i-r}| - \sum_{s' \in [s], H_{s'} \in \mathcal{H}_t} |N_{H_{s'}}(a_\ell^{H_{s'}}) \cap Y_{i-r}^{H_{s'}}| - \varepsilon^{1/5} n/r \\ &\overset{(X8)_{5.1}}{\geq} (1-d)\alpha|U_{i-r}| - \frac{2(1+d)e(\mathcal{H}_t)}{(k-1)n} - \varepsilon^{1/5} n/r \\ &\overset{(5.11)}{\geq} (1-d)\alpha|U_{i-r}| - \frac{(1+d)(1-2\nu/3)\alpha n}{Tr} - \varepsilon^{1/5} n/r \geq \alpha^2 |U_{i-r}| = \alpha^2 |V_i|. \end{aligned}$$

If $x \notin N_{H_{s+1}}(a_\ell^{H_{s+1}})$ for any $\ell \in [n_0]$, then $|B_x| \geq |V_i| - |V_i^{\mathrm{bad}}| \geq (1 - \varepsilon^{1/10})|V_i|$. So, for all $i \in [2r] \setminus [r]$ and $x \in X_i^{H_{s+1}}$, we have

$$B_x \subseteq V_i, \text{ and } |B_x| \geq \alpha^2 |V_i|. \tag{5.18}$$

For each $i \in [r]$, let $P_i := \emptyset$, and for each $i \in [2r] \setminus [r]$, let $P_i := X_i^{H_{s+1}}$. We wish to apply Lemma 3.6 with $H[Y^{H_{s+1}} \cup Z^{H_{s+1}}]$ playing the role of H and with the following objects and parameters.

object/parameter	$G(s)$	R'	V_i	P_i	$\varepsilon^{1/60}$	Δ	n'	α^2	d^3	$\mathcal{N}_{H_{s+1}}$	$f_{H_{s+1}}$	$1/(2T)$	B_x
playing the role of	G	R	V_i	X_i	ε	Δ	n	α	d	$\mathcal{M}$	f	β	A_x

Let us first check that we can indeed apply Lemma 3.6. Note that for each $ij \in E(R')$ with $i \in [2r] \setminus [r]$,

$$\begin{aligned} e_{G(s)}(V_i, V_j) &\geq e_G(V_i,V_j) - \Delta \sum_{v \in V_i} |\{s' \in [s] : v \in \phi'_s(H_{s'}[Y^{H_{s'}} \cup Z^{H_{s'}}])\}| \\ &\overset{(\Phi'3)^s_{5.1}}{\geq} e_G(V_i,V_j) - \Delta \varepsilon^{1/8} n|V_i|/r \overset{(\mathrm{A4})_{5.1}}{\geq} (1-\varepsilon^{1/9}) e_G(V_i,V_j). \end{aligned}$$

Thus (5.14) with Proposition 3.3 implies that (A1)$_{3.6}$ of Lemma 3.6 holds. Again (5.14) implies that (A2)$_{3.6}$ holds. Conditions (A3)$_{3.6}$ and (A4)$_{3.6}$ are obvious from (A1)$_{5.1}$, (X3)$_{5.1}$ and the definition of $\mathcal{N}_{H_{s+1}}$. Moreover, (5.18) implies that (A5)$_{3.6}$ also holds. Thus by Lemma 3.6, we obtain an embedding $\psi' : H_{s+1}[Y^{H_{s+1}} \cup Z^{H_{s+1}}] \to G(s)[U]$ satisfying the following.

(P1)$^{s+1}_{5.1}$ For each $x \in Y^{H_{s+1}} \cup Z^{H_{s+1}}$, we have $\psi'(x) \in B_x$,
(P2)$^{s+1}_{5.1}$ for each $e \in E(\mathcal{N}_{H_{s+1}})$, we have $|N_G(\psi'(e)) \cap V_{f_{H_{s+1}}(e)}| \geq (d^3/2)^{\Delta}|V_{f_{H_{s+1}}(e)}|$.

Let $\phi'_{s+1} := \phi_s \cup \psi \cup \psi'$. By (5.16) with the definitions of $G(s)$ and B_x, this implies $(\Phi'1)^{s+1}_{5.1}$ and $(\Phi'6)^{s+1}_{5.1}$. As $d \ll 1$, (P2)$^{s+1}_{5.1}$ implies $(\Phi'2)^{s+1}_{5.1}$, and the definitions of B_x and V_i^{bad} with (P1)$^{s+1}_{5.1}$ and $(\Phi'3)^{s}_{5.1}$ imply $(\Phi'3)^{s+1}_{5.1}$. Property (P1)$^{s+1}_{5.1}$ and (5.18) imply that $(\Phi'4)^{s+1}_{5.1}$ holds. $(\Phi'5)^{s+1}_{5.1}$ is obvious from (5.16). By repeating this for each $s \in [\kappa-1]$, we can obtain our desired packing $\phi' := \phi'_\kappa$. Since $(\Phi'1)^{\kappa}_{5.1}$–$(\Phi'5)^{\kappa}_{5.1}$ imply that ϕ' is a packing of $\mathcal{H}^\kappa$ into $G[U] \cup F'$ satisfying $(\Phi'1)_{5.1}$–$(\Phi'4)_{5.1}$, this proves the claim. □

Step 4. Packing a 3-independent set $W^H \subseteq X^H$ into $U \cup U_0$. In the previous step, we constructed a function ϕ' packing $\{H[Y^H \cup Z^H \cup A^H] : H \in \mathcal{H}\}$ into $G[U] \cup F'$. However, for each graph $H \in \mathcal{H}$, the set $\phi'(H)$ only covers a small part of U. Eventually we need to cover every vertex of G with a vertex of H. Hence, for each $H \in \mathcal{H}$ we will choose a subset $W^H \subseteq X^H$ of size exactly $|U \cup U_0| - |Y^H \cup Z^H|$, and we will construct a function ϕ'' which packs $\{H[W^H] : H \in \mathcal{H}\}$ into $G[U \cup U_0]$. As later we will extend $\phi' \cup \phi''$ into a packing of $\mathcal{H}$ into $G \cup F \cup F'$, we again have to make sure that for any $x \in X^H_i \setminus W^H$ with neighbours in W^H, there is a sufficiently large set of candidates to which x can be embedded. In other words, the set $V_i \cap N(\phi''(N_H(x) \cap W^H))$ needs to be reasonably large. To achieve this, we choose W^H to be a 3-independent set, so $|N_H(x) \cap W^H| \leq 1$, and we will map each vertex $y \in N_H(x) \cap W^H$ into a vertex v which has a large neighbourhood in V_i.

Accordingly, for all $H \in \mathcal{H}$ and $i \in [r]$, we choose a subset $W^H_i \subseteq X^H_i$ satisfying the following:

(W1)$_{5.1}$ $\bigcup_{i\in[r]} W^H_i$ is a 3-independent set of H,
(W2)$_{5.1}$ for each $i \in [r]$, we have

$$|W^H_i| = |X^H_i| - n' \overset{(\mathrm{X5})_{5.1}}{=} n_i - n' - |Y^H_i| - |Z^H_i| \pm \eta^{1/6} n \overset{(5.7),(5.6),(\mathrm{X5})_{5.1}}{=} \frac{(1 \pm \varepsilon^{1/4})n}{Tr}.$$

(W3)$_{5.1}$ $\bigcup_{i\in[r]} W^H_i \cap N^2_H(Z^H) = \emptyset$.

Indeed, the following claim ensures that there exist such sets W^H_i.

Claim 6. *For all $H \in \mathcal{H}$ and $i \in [r]$, there exists $W^H_i \subseteq X^H_i$ such that (W1)$_{5.1}$–(W3)$_{5.1}$ hold.*

Proof. We fix $H \in \mathcal{H}$. Assume that for some $i \in [r]$, we have already defined $W^H_1, \dots, W^H_{i-1}$ satisfying the following.

(W′1)$^{i-1}_{5.1}$ $\bigcup_{i'\in[i-1]} W^H_{i'}$ is a 3-independent set of H,
(W′2)$^{i-1}_{5.1}$ for each $i' \in [i-1]$, we have $|W^H_{i'}| = |X^H_{i'}| - n' = \frac{(1\pm\varepsilon^{1/4})n}{Tr}$,
(W′3)$^{i-1}_{5.1}$ $\bigcup_{i'\in[i-1]} W^H_{i'} \cap N^2_H(Z^H) = \emptyset$.

Consider $W'^H_i := X^H_i \setminus (\bigcup_{i'\in[i-1]} N^2_H(W^H_{i'}) \cup N^2_H(Z^H))$. Note that (X6)$_{5.1}$ implies that $|N^2_H(Z^H)| \leq 8\Delta^{3k^3+2}\eta^{0.9}n$. Also, (X3)$_{5.1}$ with (X6)$_{5.1}$ implies that

$$\bigcup_{i'\in[i-1]} N^2_H(W^H_{i'}) \cap X^H_i \subseteq N^1_H(Z^H) \cup \bigcup_{i' \in N_Q(i)\cap[i-1]} N^2_H(W^H_{i'}).$$

Thus

$$|W_i'^H| \geq |X_i^H| - |N_H^2(Z^H)| - \sum_{i'\in N_Q(i)\cap[i-1]} |N_H^2(W_{i'}^H)|$$
$$\overset{(\mathrm{W}'2)_{5.1}^{i-1}}{\geq} |X_i^H| - 8\Delta^{3k^3+2}\eta^{0.9}n - \frac{2k\Delta^2 n}{Tr} \overset{(\mathrm{X5})_{5.1},(5.10)}{\geq} \Delta^3(|X_i^H| - n').$$

Thus, by Lemma 3.21, $W_i'^H$ contains a 3-independent set W_i^H of size $|X_i^H| - n'$. Then, by the choice of W_i^H, $(\mathrm{W}'1)_{5.1}^i$–$(\mathrm{W}'3)_{5.1}^i$ hold. By repeating this for all $i \in [r]$ in increasing order, we obtain W_i^H satisfying $(\mathrm{W}'1)_{5.1}^r$–$(\mathrm{W}'3)_{5.1}^r$, and thus satisfying $(\mathrm{W1})_{5.1}$–$(\mathrm{W3})_{5.1}$. This proves the claim. □

For all $H \in \mathcal{H}$ and $i \in [r]$, let $W^H := \bigcup_{i'\in[r]} W_{i'}^H$ and $W_i := \bigcup_{H\in\mathcal{H}} W_i^H$, where we consider the sets $V(H)$ to be disjoint for different $H \in \mathcal{H}$. Note that for all $H \in \mathcal{H}$ and $i \in [r]$, Claim 4 implies that $0 \leq \sum_{j\in N_D^+(i)} f^H(\vec{ij}) \leq r\eta^{1/7}n$. For all $H \in \mathcal{H}$ and $i \in [r]$, we choose a partition $W_i^{H,F}, W_i^{H,U'}, W_i^{H,D}$ of W_i^H such that

$$|W_i^{H,U'}| = |U_i'| \text{ and } |W_i^{H,D}| = \sum_{j\in N_D^+(i)} f^H(\vec{ij}) \leq r\eta^{1/7}n. \tag{5.19}$$

Such partitions exist by (5.6), $(\mathrm{W2})_{5.1}$ and the fact that $\eta \ll \varepsilon \ll 1/T$. For each $S \in \{F, D, U'\}$, we let $W^{H,S} := \bigcup_{i\in[r]} W_i^{H,S}$.

We now construct a function ϕ'' which maps all the vertices of W^H into $U_0 \cup (U \setminus \phi'(Y^H \cup Z^H))$ for each $H \in \mathcal{H}$. (In Step 6 we will then apply Theorem 3.15 to embed all the vertices of $X^H \setminus W^H$ into V.) We will define ϕ'' separately for $W^{H,F}, W^{H,D}$ and $W^{H,U'}$. We first cover the 'exceptional' set U_0 with $W^{H,U'}$. (5.19) implies that for all $H \in \mathcal{H}$ and $i \in [r]$, there exists a bijection $\phi''^H_{U',i}$ from $W_i^{H,U'}$ to U_i'. We let $\phi''_{U'} := \bigcup_{H\in\mathcal{H}} \bigcup_{i\in[r]} \phi''^H_{U',i}$. Then (5.6) implies the following.

For all $i \in [r]$ and $H \in \mathcal{H}$, the function $\phi''_{U'}$ is bijective between $W_i^{H,U'}$ and U_i'. Moreover, for all $x \in W_i^{H,U'}$ and $j \in N_Q(i)$, we have $d_{G,V_j}(\phi''_{U'}(x)) \geq d^3 n'$. (5.20)

We intend to embed the neighbours of W_i^H into $\bigcup_{j\in N_Q(i)} V_j$. Thus it is natural to embed W_i^H into U_i and make use of $(\mathrm{A6})_{5.1}$. This is in fact what we will do for $W_i^{H,F}$. However, the vertices of $W_i^{H,D}$ will first be mapped to a suitable set of vertices in $U_j^D(i) \subseteq U_j$ for $j \in N_D^+(i)$. The definition of D and f^H will ensure that the remaining uncovered part of each U_j matches up exactly with the size of each $W_j^{H,F}$.

By (5.5), for all $\vec{ij} \in E(D)$ and $H \in \mathcal{H}$, we have

$$|U_j^D(i) \setminus \phi'(Y^H \cup Z^H)| \geq n/(2Tr) - |Y_j^H \cup Z_j^H| \overset{(\mathrm{X5})_{5.1},(\mathrm{X6})_{5.1}}{\geq} |U_j|/3.$$

For $i \in [r]$ and $H \in \mathcal{H}$, we let

$$b_i^H := \sum_{j\in N_D^-(i)} f^H(\vec{ji}) \overset{\text{Claim 4}}{\leq} r\eta^{1/7}n \leq \eta^{1/10}|U_i|.$$

Thus, for each $i \in [r]$, we can apply Lemma 3.18 with the following objects and parameters.

object/parameter	κ	r	$H \in \mathcal{H}$	U_i	$j \in [r]$	$U_i^D(j) \setminus \phi'(Y^H \cup Z^H)$	$\eta^{1/10}$	$f^H(\vec{ji})$	b_i^H	$1/3$
playing the role of	s	r	$i \in [s]$	A	$j \in [r]$	$A_{i,j}$	ε	$m_{i,j}$	$\sum_{j\in[r]} m_{i,j}$	d

(Recall that $f^H(\vec{ji}) = 0$ if $\vec{ji} \notin E(D)$.) Then we obtain sets $U_{i,j}^H \subseteq U_i$ satisfying the following for each $i \in [r]$, where $U_i^H := \bigcup_{j\in[r]} U_{i,j}^H$.

$(\mathrm{U1})_{5.1}$ For each $j \in [r]$ and $H \in \mathcal{H}$, we have $|U_{i,j}^H| = f^H(\vec{ji})$ and $U_{i,j}^H \subseteq U_i^D(j) \setminus \phi'(Y^H \cup Z^H)$,
$(\mathrm{U2})_{5.1}$ for $j \neq j' \in [r]$ and $H \in \mathcal{H}$, we have $U_{i,j}^H \cap U_{i,j'}^H = \emptyset$,

(U3)$_{5.1}$ for each $v \in U_i$, we have $|\{H \in \mathcal{H} : v \in U_i^H\}| \leq \eta^{1/20}|\mathcal{H}| \overset{(5.2)}{\leq} \eta^{1/20}n$.

Now for all $H \in \mathcal{H}$ and $i \in [r]$, we partition $W_i^{H,D}$ into $W_{i,1}^{H,D}, \dots, W_{i,r}^{H,D}$ in such a way that $|W_{i,j}^{H,D}| = f^H(\vec{ij})$. Clearly, this is possible by (5.19). Thus (U1)$_{5.1}$ implies that for all $(i,j) \in [r] \times [r]$ and $H \in \mathcal{H}$, we have $|W_{i,j}^{H,D}| = f^H(\vec{ij}) = |U_{j,i}^H|$. Thus there exists a bijection $\phi''^H_{D,i,j} : W_{i,j}^{H,D} \to U_{j,i}^H$. Let $\phi''_D := \bigcup_{(i,j)\in[r]\times[r]} \bigcup_{H\in\mathcal{H}} \phi''^H_{D,i,j}$. Then, for $\vec{ij} \in E(D), H \in \mathcal{H}$ and $y \in W_{i,j}^{H,D}$, we have that

$$\phi''_D(y) \in U_{j,i}^H \subseteq U_j^D(i) \setminus \phi'(Y^H \cup Z^H).$$

Thus, (5.4) with (U1)$_{5.1}$ and (U2)$_{5.1}$ implies the following.

For each $H \in \mathcal{H}$, the function ϕ''_D is bijective between $\bigcup_{i\in[r]} W_i^{H,D} = W^{H,D}$ and $\bigcup_{i\in[r]} U_i^H$. Moreover, for all $x \in W_i^{H,D}$ and $j' \in N_Q(i)$, we have $d_{G,V_{j'}}(\phi''_D(x)) \geq d^3n'/2$. (5.21)

Now, for all $H \in \mathcal{H}$ and $i \in [r]$

$$\begin{aligned} |W_i^{H,F}| &= |W_i^H| - |W_i^{H,U'}| - |W_i^{H,D}| \overset{(5.19),(W2)_{5.1}}{=} (|X_i^H| - n') - |U_i'| - \sum_{j\in N_D^+(i)} f^H(\vec{ij}) \\ &\overset{(X5)_{5.1}}{=} \widehat{n}_i^H - \sum_{j\in N_D^+(i)} f^H(\vec{ij}) - |Y_i^H| - |Z_i^H| - n' - |U_i'| \\ &\overset{\text{Claim 4}}{=} n_i - \sum_{j\in N_D^-(i)} f^H(\vec{ji}) - |Y_i^H| - |Z_i^H| - n' - |U_i'| \\ &\overset{(5.7),(U1)_{5.1}}{=} |U_i| - |Y_i^H| - |Z_i^H| - \sum_{j\in N_D^-(i)} |U_{i,j}^H| \overset{(\Phi'3)_{5.1}}{=} |U_i \setminus (\phi'(Y_i^H \cup Z_i^H) \cup U_i^H)|. \end{aligned}$$

Thus, there exists a bijection $\phi''^H_{F,i}$ from $W_i^{H,F}$ to $U_i \setminus (\phi'(Y_i^H \cup Z_i^H) \cup U_i^H)$. Let $\phi''_F := \bigcup_{H\in\mathcal{H}} \bigcup_{i\in[r]} \phi''^H_{F,i}$. Then (A6)$_{5.1}$ implies the following.

For all $H \in \mathcal{H}$ and $i \in [r]$, the function ϕ''_F is bijective between $W_i^{H,F}$ and $U_i \setminus (\phi'(Y_i^H \cup Z_i^H) \cup U_i^H)$. Moreover, for all $x \in W_i^{H,F}$ and $j \in N_Q(i)$, we have $d_{F,V_j}(\phi''_F(x)) \geq d^3n'$. (5.22)

We define

$$\phi'' := \phi''_{U'} \cup \phi''_D \cup \phi''_F \text{ and } \phi_* := \phi' \cup \phi''. \tag{5.23}$$

Then (5.20), (5.21) and (5.22) imply that ϕ'' is bijective between W^H and $(U \cup U_0) \setminus \phi'(Y^H \cup Z^H)$, when restricted to W^H for each $H \in \mathcal{H}$. Thus, we know that

$$\phi_* \text{ is bijective between } W^H \cup Y^H \cup Z^H \cup A^H \text{ and } U \cup U_0 \cup V_0 \text{ for each } H \in \mathcal{H}. \tag{5.24}$$

Moreover, (5.20), (5.21) and (5.22) imply that the following hold for all $i \in [r]$ and $H \in \mathcal{H}$.

(Φ_*1)$_{5.1}$ If $x \in W_i^{H,F}$, then $\phi_*(x) \in U$ and, for each $j \in N_Q(i)$, we have $d_{F,V_j}(\phi_*(x)) \geq d^3n'$,

(Φ_*2)$_{5.1}$ if $x \in W_i^{H,D}$, then $\phi_*(x) \in U$ and, for each $j \in N_Q(i)$, we have $d_{G,V_j}(\phi_*(x)) \geq d^3n'/2$,

(Φ_*3)$_{5.1}$ if $x \in W_i^{H,U'}$, then $\phi_*(x) \in U_0$ and, for each $j \in N_Q(i)$, we have $d_{G,V_j}(\phi_*(x)) \geq d^3n'$.

Furthermore, ($\Phi'2$)$_{5.1}$ with (U3)$_{5.1}$ implies that

(Φ_*4)$_{5.1}$ for $u \in U$, we have $|\{H \in \mathcal{H} : u \in \phi_*(Y^H \cup Z^H \cup W^{H,D})\}| \leq 2\varepsilon^{1/8}n/r$.

Step 5. Packing the graphs $H[X^H \setminus W^H]$ into internally regular graphs. Note that (X6)$_{5.1}$ and (W3)$_{5.1}$ together imply that $N_H(W^H) \cap (Y^H \cup Z^H \cup A^H) = \emptyset$ for each $H \in \mathcal{H}$. This implies that ϕ_* is a function packing $\{H[Y^H \cup Z^H \cup W^H \cup A^H] : H \in \mathcal{H}\}$ into $G[U \cup U_0] \cup F'$. We wish to pack the remaining part $H[X^H \setminus W^H]$ of each $H \in \mathcal{H}$ into $G[V]$ by using Theorem 3.15. In order to be able to apply Theorem 3.15, we first need to pack suitable subcollections of $\mathcal{H}$ into internally q-regular graphs. More precisely, for each $t \in [T]$, we will partition $\mathcal{H}_t$ into $\mathcal{H}_{t,1}, \dots, \mathcal{H}_{t,w}$ and apply Lemma 3.14 to the unembedded part of each graph in $\mathcal{H}_{t,w'}$ to pack all these parts into a graph $H_{t,w'}$ on $|V|$ vertices which is internally q-regular. We can then use Theorem 3.15 to pack all the $H_{t,w'}$ into $G[V]$ in Step 6.

For this purpose, we choose an integer q and a constant ξ such that $1/T \ll 1/q \ll \xi \ll \alpha$ and let

$$w := \frac{e(\mathcal{H})}{(1-3\xi)T(k-1)qn/2} \overset{(\mathrm{A2})_{5.1}}{\leq} \frac{(1-\nu/2)\alpha n'}{qT}. \tag{5.25}$$

By using (5.1) and (5.11), for each $t \in [T]$, we can further partition $\mathcal{H}_t$ into $\mathcal{H}_{t,1}, \dots, \mathcal{H}_{t,w}$ such that for each $(t, w') \in [T] \times [w]$, we have

$$e(\mathcal{H}_{t,w'}) = (1-3\xi)(k-1)qn/2 \pm 2\Delta n = (1-3\xi \pm \xi/2)(k-1)qn/2. \tag{5.26}$$

By $(\mathrm{A1})_{5.1}$, we have

$$|\mathcal{H}_{t,w'}| \leq 2(k-1)q \leq (q\xi)^{3/2}. \tag{5.27}$$

For all $H \in \mathcal{H}$ and $i \in [r]$, let $\widetilde{X}_i^H := X_i^H \setminus W_i^H$ and $\widetilde{X}^H := \bigcup_{i\in[r]} \widetilde{X}_i^H$. Thus, by $(\mathrm{W2})_{5.1}$ we have $|\widetilde{X}_i^H| = n'$ for all $H \in \mathcal{H}$ and $i \in [r]$. Moreover, for all $t \in [T]$, $w' \in [w]$ and $ij \in E(Q)$, we have

$$\begin{aligned}\sum_{H\in\mathcal{H}_{t,w'}} e(H[\widetilde{X}_i^H, \widetilde{X}_j^H]) &= \sum_{H\in\mathcal{H}_{t,w'}} (e(H[X_i^H, X_j^H]) \pm \Delta(|W_i^H| + |W_j^H|) \\ &\overset{(\mathrm{X4})_{5.1},(\mathrm{W2})_{5.1}}{=} \sum_{H\in\mathcal{H}_{t,w'}} \left(\frac{2e(H) \pm \varepsilon^{1/5} n}{(k-1)r} \pm \frac{3\Delta n}{Tr}\right) \overset{(5.26)}{=} (1-3\xi\pm\xi)qn'\end{aligned} \tag{5.28}$$

When packing $H[\tilde{X}^H]$ and $H'[\tilde{X}^{H'}]$ (say) into the same graph $H_{t,w'}$, we need to make sure that the 'attachment sets' of $H[\tilde{X}^H]$ and $H'[\tilde{X}^{H'}]$ are not mapped to the same vertex sets in $H_{t,w'}$. The attachment set for $H[\tilde{X}^H]$ contains those vertices of $\tilde{X}^H$ which have a neighbour in $W^H \cup Y^H \cup Z^H \cup A^H$ (more precisely, a neighbour in $W^H \cup Z^H$) and is defined in (5.29). Keeping these attachment sets disjoint in $H_{t,w'}$ ensures that we can make the embedding of each $\tilde{X}^H$ consistent with the existing partial embedding of H without attempting to use an edge of F or G twice. For all $i \in [r]$ and $H \in \mathcal{H}$, we let

$$N_i^{H,F} := \bigcup_{i'\in N_Q(i)} N_H^1(W_{i'}^{H,F}) \cap \widetilde{X}_i^H \quad \text{and } N_i^{H,G} := N_H^1(Z^H \cup W^{H,D} \cup \bigcup_{i'\in N_Q(i)} W_{i'}^{H,U'}) \cap \widetilde{X}_i^H. \tag{5.29}$$

Note that $(\mathrm{W1})_{5.1}$, $(\mathrm{W3})_{5.1}$ and the fact that $W^{H,F}, W^{H,D}, W^{H,U'}$ form a partition of W^H implies that

$$N_i^{H,F} \cap N_i^{H,G} = \emptyset. \tag{5.30}$$

Moreover, if $x \in N_i^{H,F}$ then x has a unique neighbour in $W^{H,F}$. Similarly, if $x \in N_i^{H,G}$, then either x has a unique neighbour in $W^{H,D} \cup W^{H,U'}$ or x has at least one neighbour in Z^H (but not both). Note that for $i \in [r]$ and $H \in \mathcal{H}$,

$$\begin{aligned}|N_i^{H,F} \cup N_i^{H,G}| &\leq \sum_{i'\in N_Q(i)} \Delta(|W_{i'}^{H,F}| + |W_{i'}^{H,U'}|) + \Delta(|Z^H| + |W^{H,D}|) \\ &\overset{\substack{(\mathrm{X6})_{5.1},\\ (\mathrm{W2})_{5.1},(5.19)}}{\leq} \frac{2\Delta kn}{Tr} + 4\Delta^{3k^3+1}\eta^{0.9}n + \Delta r^2\eta^{1/7}n \leq T^{-2/3}n'.\end{aligned} \tag{5.31}$$

For each $i \in [r]$, we consider a set $\widehat{X}_i$ with $|\widehat{X}_i| = n'$ such that $\widehat{X}_1, \dots, \widehat{X}_r$ are pairwise vertex-disjoint. For each $(t, w') \in [T] \times [w]$, let $\mathcal{H}_{t,w'} =: \{H_{t,w'}^1, \dots, H_{t,w'}^{h(t,w')}\}$. Then, by (5.27), (5.28), (5.31) and $(\mathrm{X3})_{5.1}$, we can apply Lemma 3.14 with the following objects and parameters for each $(t, w') \in [T] \times [w]$.

object/parameter	$H^j_{t,w'}[\widetilde{X}^{H^j_{t,w'}}]$	$\widetilde{X}_i^{H^j_{t,w'}}$	$\widehat{X}_i$	n'	q	ξ	$T^{-2/3}$	$h(t,w')$	$N_i^{H^j_{t,w'},F} \cup N_i^{H^j_{t,w'},G}$	Q
playing the role of	L_j	X_i^j	V_i	n	q	ξ	ε	s	W_i^j	R

Then for each $(t,w') \in [T] \times [w]$, we obtain a function $\Phi_{t,w'}$ packing $\{H[\widetilde{X}^H] : H \in \mathcal{H}_{t,w'}\}$ into some graph $H_{t,w'}$ which is internally q-regular with respect to the vertex partition $(Q, \widehat{X}_1, \dots, \widehat{X}_r)$. Moreover, for all $i \in [r]$ and $H \in \mathcal{H}_{t,w'}$ we have $\Phi_{t,w'}(\widetilde{X}_i^H) = \widehat{X}_i$ and for distinct $H, H' \in \mathcal{H}_{t,w'}$ and $i \in [r]$, we have

$$\Phi_{t,w'}(N_i^{H,F} \cup N_i^{H,G}) \cap \Phi_{t,w'}(N_i^{H',F} \cup N_i^{H',G}) = \emptyset. \tag{5.32}$$

Note that for all $(t,w') \in [T] \times [w]$, the graphs $H_{t,w'}$ have same vertex set $\bigcup_{i\in[r]} \widehat{X}_i$. For all $i \in [r]$ and $(t,w') \in [T] \times [w]$, we let

$$L_i^{t,w'} := \bigcup_{H\in\mathcal{H}_{t,w'}} \Phi_{t,w'}(N_i^{H,F}) \quad \text{and} \quad M_i^{t,w'} := \bigcup_{H\in\mathcal{H}_{t,w'}} \Phi_{t,w'}(N_i^{H,G}). \tag{5.33}$$

Then by (5.30) and (5.32) we have

$$L_i^{t,w'} \cup M_i^{t,w'} \subseteq \widehat{X}_i \quad \text{and} \quad L_i^{t,w'} \cap M_i^{t,w'} = \emptyset. \tag{5.34}$$

By (5.27) and (5.31), for all $(t,w') \in [T] \times [w]$ and $i \in [r]$

$$|L_i^{t,w'} \cup M_i^{t,w'}| \le q^{3/2} T^{-2/3} n' \le T^{-1/2} n'. \tag{5.35}$$

Step 6. Packing the internally regular graphs $H_{t,w'}$ into $G[V]$. In the previous step, we constructed a collection $\widehat{\mathcal{H}} := \{H_{1,1}, \dots, H_{T,w}\}$ of internally q-regular graphs on $|V|$ vertices. We now wish to apply Theorem 3.15 to pack $\widehat{\mathcal{H}}$ into $G[V]$. However, our packing needs to be consistent with the packing ϕ_*. Note that for each $H \in \mathcal{H}$ the set $W^H \cup Y^H \cup Z^H \cup A^H$ consists of exactly those vertices of H which are already embedded by ϕ_*. Thus by (X3)$_{5.1}$, (X6)$_{5.1}$, (5.29) and (5.33), it follows that whenever $x \in \widehat{X}_i$ is a vertex of $H_{t,w'}$ such that the set $\Phi^{-1}_{t,w'}(x)$ of pre-images of x contains a neighbour of some vertex which is already embedded by ϕ_*, then $x \in L_i^{t,w'} \cup M_i^{t,w'}$. Thus in order to ensure that our packing of $\widehat{\mathcal{H}}$ is consistent with ϕ_*, for each $i \in [r]$, each $(t,w') \in [T] \times [w]$ and each $y \in L_i^{t,w'} \cup M_i^{t,w'}$ we will choose a suitable target set $A_y^{t,w'}$ of vertices of $G[V]$ and will map y into this set.

For all $(t,w') \in [T] \times [w]$, $i \in [r]$ and any vertex $y \in L_i^{t,w'} \cup M_i^{t,w'}$, (5.32) implies that there exists a unique graph $H_y^{t,w'} \in \mathcal{H}_{t,w'}$ and a unique vertex $x_y^{t,w'} \in N_i^{H_y^{t,w'},F} \cup N_i^{H_y^{t,w'},G}$ such that $y = \Phi_{t,w'}(x_y^{t,w'})$. Let

$$J_y^{t,w'} := N_{H_y^{t,w'}}(x_y^{t,w'}) \cap (W^{H_y^{t,w'}} \cup Z^{H_y^{t,w'}}) = N_{H_y^{t,w'}}(x_y^{t,w'}) \cap (W^{H_y^{t,w'}} \cup Y^{H_y^{t,w'}} \cup Z^{H_y^{t,w'}} \cup A^{H_y^{t,w'}}).$$

The final equality follows from (X6)$_{5.1}$. For all $(t,w') \in [T] \times [w]$, $i \in [r]$ and any vertex $y \in L_i^{t,w'} \cup M_i^{t,w'}$, we define the target set

$$A_y^{t,w'} := \begin{cases} N_F(\phi_*(J_y^{t,w'})) \cap V_i & \text{if } x_y^{t,w'} \in N_i^{H_y^{t,w'},F}, \\ N_G(\phi_*(J_y^{t,w'})) \cap V_i & \text{if } x_y^{t,w'} \in N_i^{H_y^{t,w'},G}. \end{cases}$$

Note that $A_y^{t,w'}$ is well-defined as (5.30) implies that exactly one of the above cases holds. Moreover, the following claim implies that these target sets are sufficiently large.

Claim 7. *For all $(t,w') \in [T] \times [w]$, $i \in [r]$ and any vertex $y \in L_i^{t,w'} \cup M_i^{t,w'}$, we have*

$$|A_y^{t,w'}| \ge d^{5\Delta} |V_i|.$$

Proof. We fix $(t,w') \in [T]\times[w]$, $i\in[r]$ and a vertex $y \in L_i^{t,w'} \cup M_i^{t,w'}$. For simplicity, we write $H := H_y^{t,w'}$, $x := x_y^{t,w'}$ and $J := J_y^{t,w'}$. Then (5.30) implies that exactly one of the following two cases holds.

Case 1. $x \in N_i^{H,F}$. In this case, (W1)$_{5.1}$ and (W3)$_{5.1}$ imply that

$$J = N_H(x) \cap W^{H,F} \overset{(\mathrm{X3})_{5.1}}{=} N_H(x) \cap \bigcup_{i' \in N_Q(i)} W_{i'}^{H,F} \quad \text{and} \quad |J| = 1.$$

Then by $(\Phi_*1)_{5.1}$, we know $|A_y^{t,w'}| \geq d^3|V_i|$.

Case 2. $x \in N_i^{H,G}$. In this case, by (5.29) and (W3)$_{5.1}$, we have exactly one of the following cases.

Case 2.1 $x \in N_H^1(Z^H)$. In this case, $N_H(x) \cap W^H = \emptyset$ by (W3)$_{5.1}$. Thus we have $J = N_H(x) \cap Z^H$. Then (5.15) and $(\Phi'1)_{5.1}$ imply that $|A_y^{t,w'}| = |N_G(\phi'(e_{H,x})) \cap V_{f_H(e_{H,x})}| \geq d^{5\Delta}|V_i|$.

Case 2.2 $x \in N_H^1(W^{H,D} \cup W^{H,U'})$. In this case, again (W1)$_{5.1}$, (W3)$_{5.1}$ and (X3)$_{5.1}$ imply that

$$J = N_H(x) \cap (W^{H,D} \cup W^{H,U'}) = N_H(x) \cap \bigcup_{i' \in N_Q(i)} (W_{i'}^{H,D} \cup W_{i'}^{H,U'}) \quad \text{and} \quad |J| = 1.$$

Thus $(\Phi_*2)_{5.1}$ or $(\Phi_*3)_{5.1}$ imply that $|A_y^{t,w'}| \geq d^3|V_i|/2$. This proves the claim. □

Let $\mathbf{S} := [T]\times[w]$. Let Λ be the graph with

$$V(\Lambda) := \{(\vec{s}, y) : \vec{s} \in \mathbf{S}, y \in \bigcup_{\vec{s}\in\mathbf{S}, i\in[r]} L_i^{\vec{s}} \cup M_i^{\vec{s}}\}$$

and

$$E(\Lambda) := \left\{(\vec{s},y)(\vec{t},y') : \vec{s} \neq \vec{t} \in \mathbf{S}, i\in[r], (y,y') \in (L_i^{\vec{s}} \times L_i^{\vec{t}}) \cup (M_i^{\vec{s}} \times M_i^{\vec{t}}) \text{ and } \phi_*(J_y^{\vec{s}}) \cap \phi_*(J_{y'}^{\vec{t}}) \neq \emptyset\right\}.$$

Note that Λ is the graph indicating possible overlaps of images of distinct edges when we extend ϕ_*. Indeed, if $(\vec{s},y)$ and $(\vec{t},y')$ are adjacent in Λ, there are $z \in N_{H_y^{\vec{s}}}(x_y^{\vec{s}})$ and $z' \in N_{H_{y'}^{\vec{t}}}(x_{y'}^{\vec{t}})$ such that $\phi_*(z) = \phi_*(z')$. If we embed y and y' onto the same vertex, then the two edges $x_y^{\vec{s}}z$ and $x_{y'}^{\vec{t}}z'$ would be embedded onto the same edge of $G \cup F$. Thus we need to ensure that $\phi(y) \neq \phi(y')$.

Note that for all $(\vec{s},y) \in V(\Lambda)$ and $\vec{t} \in \mathbf{S}$, we have

$$\begin{aligned}
|\{(\vec{t},y') \in N_\Lambda((\vec{s},y))\}| &\leq |\{y' : H_{y'}^{\vec{t}} \in \mathcal{H}_{\vec{t}}, \phi_*(J_{y'}^{\vec{t}}) \cap \phi_*(J_y^{\vec{s}}) \neq \emptyset\}| \\
&\leq \sum_{v \in \phi_*(J_y^{\vec{s}})} |\{y' : H_{y'}^{\vec{t}} \in \mathcal{H}_{\vec{t}}, v \in \phi_*(J_{y'}^{\vec{t}})\}| \\
&\leq \sum_{v \in \phi_*(J_y^{\vec{s}})} \sum_{H \in \mathcal{H}_{\vec{t}}} |\{x \in V(H) : v \in \phi_*(N_H(x))\}| \\
&\leq \sum_{v \in \phi_*(J_y^{\vec{s}})} \sum_{H \in \mathcal{H}_{\vec{t}}} |\{x \in N_H(x') : v = \phi_*(x'), x' \in V(H)\}| \\
&\overset{(5.24)}{\leq} \sum_{v \in \phi_*(J_y^{\vec{s}})} \sum_{H \in \mathcal{H}_{\vec{t}}} \Delta \leq \Delta^2 |\mathcal{H}_{\vec{t}}| \overset{(5.27)}{\leq} \Delta^2 (\xi q)^{3/2} \leq q^2.
\end{aligned} \tag{5.36}$$

(Here the third inequality holds by the definition of $J_{y'}^{\vec{t}}$ and the definition of $x_{y'}^{\vec{t}}$, the fifth inequality holds since (5.24) implies that there is at most one $x' \in V(H)$ with $\phi_*(x') = v$, and the sixth inequality holds since $|J_y^{\vec{s}}| \leq |N_{H_y^{\vec{s}}}(x_y^{\vec{s}})| \leq \Delta$.)

Consider any $(\vec{s},y) \in V(\Lambda)$. Then similarly as above we have

$$d_\Lambda((\vec{s},y)) \leq \sum_{v \in \phi_*(J_y^{\vec{s}})} \sum_{H \in \mathcal{H}} |\{x \in N_H(x') : v = \phi_*(x'), x' \in V(H)\}| \leq \Delta^2 |\mathcal{H}| \overset{(5.2)}{\leq} \alpha^{1/2} n'.$$

This shows that

$$\Delta(\Lambda) \leq \alpha^{1/2} n' < d^{5\Delta} n'/2. \tag{5.37}$$

We can now apply the blow-up lemma for approximate decompositions (Theorem 3.15) with the following objects and parameters.

object/parameter	$G[V]$	V_i	$\widehat{X}_i$	$H_{t,w'}$	$\mathbf{S}=[T]\times[w]$	q	$T^{-1/2}$	Q
playing the role of	G	V_i	X_i	H_i	$[s]$	q	ε	R
object/parameter	r	$L_i^{t,w'} \cup M_i^{t,w'}$	$A_y^{t,w'}$	α	$d^{5\Delta}$	ν	Λ	n'
playing the role of	r	W_i^j	A_w^j	d	d_0	α	Λ	n

Indeed, $(A3)_{5.1}$ implies that $(A1)_{3.15}$ holds, and $(A2)_{3.15}$ holds by the definition of $H_{t,w'}$. Claim 7 and (5.35) imply that $(A3)_{3.15}$ holds, and (5.35), (5.36) and (5.37) imply that $(A4)_{3.15}$ holds. Moreover, (5.25) implies that the upper bound on s in the assumption of Theorem 3.15 holds.

Thus by Theorem 3.15 we obtain a function ϕ^* that packs $\{H_{\vec{s}} : \vec{s} \in \mathbf{S}\}$ into $G[V]$ and satisfies the following, where $\phi^*_{\vec{s}}$ denotes the restriction of ϕ^* to $H_{\vec{s}}$.

$(\Phi^*1)_{5.1}$ for each $\vec{s} \in \mathbf{S}$ and $y \in \bigcup_{i\in[r]} L_i^{\vec{s}} \cup M_i^{\vec{s}}$, we have $\phi^*_{\vec{s}}(y) \in A_y^{\vec{s}}$,

$(\Phi^*2)_{5.1}$ for any $(\vec{s},y)(\vec{t},y') \in E(\Lambda)$, we have that $\phi^*_{\vec{s}}(y) \neq \phi^*_{\vec{t}}(y')$.

We let

$$\phi := \phi^*(\bigcup_{\vec{s}\in\mathbf{S}} \Phi_s) \cup \phi_*.$$

Recall from Step 3 and (5.23) that $\phi_* = \phi' \cup \phi''$, and that ϕ' packs $\{H[Y^H \cup Z^H \cup A^H] : H \in \mathcal{H}\}$ into $G[U] \cup F'$. Since each $\Phi_{\vec{s}}$ is a packing of $\{H[X^H \setminus W^H] : H \in \mathcal{H}_{\vec{s}}\}$ into $H_{\vec{s}}$ and ϕ^* is a packing of $\{H_{\vec{s}} : \vec{s} \in \mathbf{S}\}$ into $G[V]$, we know that ϕ packs $\{H[X^H \setminus W^H] : H \in \mathcal{H}\}$ into $G[V]$. Moreover, $(\Phi^*1)_{5.1}$, $(\Phi^*2)_{5.1}$ with the definitions of $A_y^{\vec{s}}$ and Λ imply that ϕ packs $\{H[X^H \setminus W^H, W^{H,F}] : H \in \mathcal{H}\}$ into F, and ϕ packs $\{H[X^H \setminus W^H, W^{H,U'}] : H \in \mathcal{H}\}$ into $G[V,U_0]$, and ϕ packs $\{H[X^H \setminus W^H, W^{H,D} \cup Z^H] : H \in \mathcal{H}\}$ into $G[V,U]$. Thus, we have the following.

$$\begin{aligned}
&\phi(\bigcup_{H\in\mathcal{H}} E_H(Y^H \cup Z^H \cup A^H)) \subseteq E_G(U) \cup E(F'), \qquad \phi(\bigcup_{H\in\mathcal{H}} E_H(X^H \setminus W^H)) \subseteq E_G(V),\\
&\phi(\bigcup_{H\in\mathcal{H}} E_H(X^H \setminus W^H, W^{H,F})) \subseteq E_F(V,U), \qquad \phi(\bigcup_{H\in\mathcal{H}} E_H(X^H \setminus W^H, W^{H,U'})) \subseteq E_G(V,U_0),\\
&\phi(\bigcup_{H\in\mathcal{H}} E_H(X^H \setminus W^H, W^{H,D} \cup Z^H)) \subseteq E_G(V,U).
\end{aligned} \tag{5.38}$$

Also, it is obvious that the restriction of ϕ to $V(H)$ is injective for each $H \in \mathcal{H}$. As $G[U] \cup F', G[V], F, G[V,U_0]$ and $G[V,U]$ are pairwise edge-disjoint, we conclude that ϕ packs $\mathcal{H}$ into $G \cup F \cup F'$. Moreover, by (5.2) we have $\Delta(\phi(\mathcal{H})) \leq \Delta|\mathcal{H}| \leq 4k\Delta\alpha n/r$, thus $(B1)_{5.1}$ holds. By (5.38) and $(\Phi'4)_{5.1}$, for $u \in U$, we have

$$d_{\phi(\mathcal{H})\cap G}(u) \leq \Delta|\{H \in \mathcal{H} : u \in \phi_*(Y^H \cup Z^H \cup W^{H,D})\}| \overset{(\Phi_*4)_{5.1}}{\leq} \frac{2\Delta\varepsilon^{1/8}n}{r}.$$

Thus $(B2)_{5.1}$ holds.

Finally, for $i \in [r]$, by $(X3)_{5.1}$, $(X6)_{5.1}$, (5.38) we have

$$\begin{aligned}
e_{\phi(\mathcal{H})\cap G}(V_i, U \cup U_0) &\leq \sum_{H\in\mathcal{H}} \Delta\Big(|Z^H| + \sum_{j\in N_Q(i)} |W_j^{H,U'}| + \sum_{j\in N_Q(i)} |W_j^{H,D}|\Big)\\
&\overset{(5.2),(X6)_{5.1},\,(5.6),(5.19)}{\leq} \frac{2k\Delta\alpha n}{r}\Big(4\Delta^{3k^3}\eta^{0.9}n + 2(k-1)\varepsilon^{3/4}n/r + (k-1)r\eta^{1/7}n\Big)\\
&\leq \frac{\varepsilon^{1/2}n^2}{r^2},
\end{aligned}$$

which shows that $(B3)_{5.1}$ holds.

□

6. Proof of Theorem 1.2

The proof of Theorem 1.2 proceeds in three steps. In the first step we will apply the results of Section 3 to construct suitable edge-disjoint subgraphs $G_{t,s}, G^*_t, F_{t,s}$ and F'_t of G, where $G_{t,s}$ is a K_k-factor blow-up spanning almost all vertices while $G^*_t, F_{t,s}$ and F'_t are comparatively sparse. In the (straightforward) second step, we simply partition $\mathcal{H}$ into collections $\mathcal{H}_{t,s}$ such that the $e(\mathcal{H}_{t,s})$ are approximately equal to each other. Finally, in the third step we will pack each $\mathcal{H}_{t,s}$ into $G_{t,s} \cup G^*_t \cup F_{t,s} \cup F'_t$ via Lemma 5.1.

Proof of Theorem 1.2. Let $\sigma := \delta - \max\{1/2, \delta_k^{\rm reg}\} > 0$. By (3.6), we have $\delta \geq 1 - 1/k + \sigma$ for any $k \geq 2$. Without loss of generality, we may assume that $\nu < \sigma/2$. For given $\nu, \sigma > 0$ and Δ, $k \in \mathbb{N}\backslash\{1\}$, we choose constants n_0, ξ, η, M, M', ε, T, q, d such that $q \mid T$ and

$$0 < 1/n_0 \ll \eta \ll 1/M \ll 1/M' \ll \varepsilon \ll 1/T \ll 1/q \ll \xi \ll d \ll \nu, \sigma, 1/\Delta, 1/k \leq 1/2. \tag{6.1}$$

Suppose $n \geq n_0$ and let G be an n-vertex graph satisfying condition (i) of Theorem 1.2. Furthermore, suppose $\mathcal{H}$ is a collection of (k,η)-chromatic η^2-separable graphs satisfying conditions (ii) and (iii) of Theorem 1.2. We will show that $\mathcal{H}$ packs into G. Note that we assume $\mathcal{H}$ to consist of η^2-separable graphs here (instead of η-separable graphs). This is more convenient for our purposes, but still implies Theorem 1.2.

Step 1. Decomposing G into host graphs. In this step, we apply Szemerédi's regularity lemma to G and then apply Lemma 3.16 to obtain a partition of $V(G)\backslash V_0$ into T reservoir sets Res_t, where V_0 is the exceptional set obtained from Szemerédi's regularity lemma. We use Lemma 3.13 to obtain an approximate decomposition of the reduced multi-graph $R'_{\rm multi}$ of G into almost K_k-factors and partition these factors into T collections. Each such almost K_k-factor Q gives us an ε-regular Q-blow-up $G_{t,s}$ in G, and we modify it into a super-regular Q-blow-up. We also put aside several sparse 'connection graphs' $F_{t,s}$ and F'_t, which will be used to link vertices in the reservoir and exceptional set with vertices in the rest of the graph. These connection graphs will play the roles of F and F' in Lemma 5.1. We also put aside a further sparse connection graph G^*_t which provides additional connections within $V(G) \setminus V_0$.

We apply Szemerédi's regularity lemma (Lemma 3.5) with (ε^2, d) playing the role of (ε, d) to obtain a partition $V'_0, \dots, V'_{r'}$ of $V(G)$ and a spanning subgraph $G' \subseteq G$ such that

(R1) $M' \leq r' \leq M$,
(R2) $|V'_0| \leq \varepsilon^2 n$,
(R3) $|V'_i| = |V'_j| = (1 \pm \varepsilon^2)n/r'$ for all $i, j \in [r']$,
(R4) for all $v \in V(G)$ we have $d_{G'}(v) > d_G(v) - 2dn$,
(R5) $e(G'[V'_i]) = 0$ for all $i \in [r']$,
(R6) for any i, j with $1 \leq i \leq j \leq r'$, the graph $G'[V'_i, V'_j]$ is either empty or $(\varepsilon^2, d_{i,j})$-regular for some $d_{i,j} \in [d, 1]$.

Let R' be the graph with

$$V(R') = [r'] \qquad \text{and} \qquad E(R') := \{ij : e_{G'}(V'_i, V'_j) > 0\}.$$

Note that for $i, j \in [r']$, $ij \in E(R')$ if and only if $G'[V'_i, V'_j]$ is $(\varepsilon^2, d_{i,j})$-regular with $d_{i,j} \geq d$. Now, we let $R'_{\rm multi}$ be a multi-graph with $V(R'_{\rm multi}) = [r']$ and with exactly

$$q_{i,j} := \lfloor (1 - 6d) d_{i,j} q \rfloor \tag{6.2}$$

edges between i and j for each $ij \in E(R')$. Note that $R'_{\rm multi}$ has edge-multiplicity at most q. For each $i \in [r']$, we have

$$\begin{aligned} d_{R'_{\rm multi}}(i) &= \sum_{j \in N_{R'}(i)} \lfloor (1-6d) q (\frac{e_{G'}(V'_i, V'_j)}{|V'_i||V'_j|} \pm \varepsilon^2) \rfloor \overset{\text{(R3),(R5)}}{=} \frac{\sum_{v \in V'_i} (1-6d) q d_{G', V(G)\setminus V_0}(v)}{|V'_i|^2} \pm \varepsilon^2 q r' \pm r' \\ &\overset{\text{(R2),(R4)}}{=} \frac{q}{|V'_i|^2} \sum_{v \in V'_i} (d_G(v) \pm 10dn) \pm 2r' \overset{(i)}{=} \frac{(\delta \pm 11d)qn}{|V'_i|} \pm 2r' \overset{\text{(R3)}}{=} (\delta \pm d^{3/4}) q r'. \end{aligned} \tag{6.3}$$

We apply Lemma 3.13 with $R'_{\rm multi}, r', \varepsilon^2, k, \sigma, d^{3/4}, \nu/5, T$ and q playing the roles of $G,n,\varepsilon,k,\sigma,\xi,\nu,T$ and q, respectively. Then, by permuting indices in $[r']$ if necessary, we obtain $R_{\rm multi} \subseteq R'_{\rm multi}$ and a collection $\mathcal{Q} := \{Q_{1,1},\dots,Q_{1,\kappa/T},Q_{2,1},\dots,Q_{T,\kappa/T}\}$ of edge-disjoint subgraphs of $R_{\rm multi}$ such that the following hold.

(Q1) $R_{\rm multi} = R'_{\rm multi}[[r]]$ with $(1-\varepsilon^2)r' \le r \le r'$, and $k \mid r$,
(Q2) $\kappa = \frac{(\delta-\nu/5\pm\varepsilon^2)qr'}{k-1} = \frac{(\delta-\nu/5\pm\varepsilon)qr}{k-1}$ and $T \mid \kappa$,
(Q3) for each $(t,s) \in [T]\times[\kappa/T]$, $Q_{t,s}$ is a vertex-disjoint union of at least $(1-\varepsilon)r/k$ copies of K_k,
(Q4) for each $i \in [r]$, we have $|\{(t,s)\in[T]\times[\kappa/T] : i \in V(Q_{t,s})\}| \ge \kappa - \varepsilon r$.
(Q5) for all $t \in [T]$ and $i,j \in [r]$, we have $|\{s \in [\kappa/T] : j \in N_{Q_{t,s}}(i)\}| \le 1$.

For each $t \in [T]$, let $\mathcal{Q}_t := \{Q_{t,1},\dots,Q_{t,\kappa/T}\}$. We define $R := R'[[r]]$ to be the induced subgraph of R' on $[r]$. Note that each $Q_{t,s} \in \mathcal{Q}$ can be viewed as a subgraph of R. Moreover, for fixed $t \in [T]$, (Q5) implies that the graphs $Q_{t,1},\dots,Q_{t,\kappa/T}$ are pairwise edge-disjoint when viewed as subgraphs of R. Also, we have

$$\delta(R) \ge q^{-1}\delta(R'_{\rm multi}) - (r'-r) \overset{(6.3),(Q1)}{\ge} (\delta - d^{1/2})r. \tag{6.4}$$

We need to modify the sets V'_i later to ensure that we obtain appropriate super-regular $Q_{t,s}$-blow-ups. For this, we need to move some 'bad' vertices in V'_i into V'_0. For each $i \in [r]$ and each $j \in N_R(i)$, we define

$$U_i(j) := \{v \in V'_i : d_{G',V'_j}(v) \ne (d_{i,j} \pm \varepsilon^2)|V'_j|\} \text{ and } U'_i := \{v \in V'_i : |\{j : v \in U_i(j)\}| > \varepsilon r\}. \tag{6.5}$$

By Proposition 3.4 and (R6), for any $i \in [r]$ and $j \in N_R(i)$ we have

$$|U_i(j)| \le 5\varepsilon^2 n/r \qquad \text{and} \qquad |U'_i| \le (\varepsilon r)^{-1} \sum_{j \in N_R(i)} |U_i(j)| \le 5\varepsilon n/r. \tag{6.6}$$

For each $i \in [r]$, we let $V_i := V'_i \setminus U'_i$ and $V_0 := V'_0 \cup \bigcup_{i=1}^r U'_i \cup \bigcup_{i=r+1}^{r'} V'_i$.

By (R2) and (R3), for each $i \in [r]$, we have

$$(1-6\varepsilon)n/r \le |V_i| \le n/r \qquad \text{and} \qquad |V_0| \le 6\varepsilon n. \tag{6.7}$$

We apply Lemma 3.16 with $G', V(G)\setminus V_0, \{V_i\}_{i=1}^r$ and T playing the roles of $G, V, \{V_i\}_{i=1}^r$ and t to obtain a partition $\{Res_1,\dots,Res_T\}$ of $V(G)\setminus V_0$ satisfying the following, where we define $V_i^t := V_i \cap Res_t$.

(Res1) For all $t \in [T]$ and $v \in V(G)$, we have $d_{G',V_i^t}(v) = \frac{1}{T}d_{G',V_i}(v) \pm n^{2/3}$,
(Res2) for all $t \in [T]$ and $i \in [r]$, we have $|V_i^t| = (\frac{1}{T} \pm \varepsilon^2)|V_i| \overset{(6.7)}{=} \frac{(1\pm7\varepsilon)n}{Tr}$,
(Res3) for all $t \in [T]$, we have $|Res_t| \in \{\lfloor \frac{n-|V_0|}{T} \rfloor, \lfloor \frac{n-|V_0|}{T} \rfloor + 1\}$.

Next, we partition the edges in $G' \setminus V_0$ into $L_1,\dots,L_7$ which will be the building blocks for the graphs G, F and F' in Lemma 5.1. Let $p_1 := 1-6d$ and $p_j := d$ for $2 \le j \le 7$. Apply Lemma 3.17 with $G' \setminus V_0$, $\{V_i^t : i \in [r], t \in [T]\}$, $\{(V_i,V_j) : ij \in E(R)\}$ and 7 playing the roles of $G, \mathcal{U}, \mathcal{U}'$ and s. Then we obtain a decomposition $L_1,\dots,L_7$ of $G'\setminus V_0$ satisfying the following for all $t \in [T]$, $i \in [r]$, $\ell \in [7]$ and $v \in V(G)\setminus V_0$:

(L1) $d_{L_\ell,V_i^t}(v) = p_\ell d_{G',V_i^t}(v) \pm n^{2/3}$,
(L2) for each $ij \in E(R)$, we have that $L_\ell[V_i,V_j]$ is $(4\varepsilon^2, d_{i,j}p_\ell)$-regular.

Let $G'' := L_1$. For each $t \in [T]$, let G_t^*, F_t and F_t^* be the graphs on vertex set $V(G)\setminus V_0$ with

$$E(G_t^*) := \bigcup_{t'=1}^{t-1} E(L_2[Res_t, Res_{t'}]) \cup \bigcup_{t'=t+1}^{T} E(L_3[Res_t, Res_{t'}]) \cup L_2[Res_t], \tag{6.8}$$
$$E(F_t) := \bigcup_{t'=1}^{t-1} E(L_4[Res_t, Res_{t'}]) \cup \bigcup_{t'=t+1}^{T} E(L_5[Res_t, Res_{t'}]),$$

$$E(F_t^*) := \bigcup_{t'=1}^{t-1} E(L_6[Res_t, Res_{t'}]) \cup \bigcup_{t'=t+1}^{T} E(L_7[Res_t, Res_{t'}]).$$

For each $t \in [T]$, we let $F_{t,1}, \dots, F_{t,\kappa/T}$ be subgraphs of F_t such that for all $s \in [\kappa/T]$

$$F_{t,s} := \bigcup_{i \in V(Q_{t,s})} \bigcup_{j \in N_{Q_{t,s}}(i)} F_t[V_i^t, V_j \setminus Res_t]. \tag{6.9}$$

Note that (Q5) implies that for $s \neq s' \in [\kappa/T]$, the graphs $F_{t,s}$ and $F_{t,s'}$ are edge-disjoint. Thus $G'', G_1^*, \dots, G_T^*, F_{1,1}, \dots, F_{T,\kappa/T}, F_1^*, \dots, F_T^*$ form edge-disjoint subgraphs of $G' \setminus V_0$. The edges in G_t^* will be used to satisfy condition (A4)$_{5.1}$ when applying Lemma 5.1. The graphs $F_{t,s}$ will play the role of F in Lemma 5.1. The graphs F_t^* will be used in the construction of the graph F_t', which will play the role of F' in Lemma 5.1.

We will now further partition the edges in $G'' = L_1$. Note that for each $ij \in E(R)$, by (6.2) we have $q_{i,j} = \lfloor d_{i,j} p_1 q \rfloor$. To further partition G'', we apply Lemma 3.17 for each $ij \in E(R)$ with the following objects and parameters.

object/parameter	$G''[V_i, V_j]$	$\{V_i^t, V_j^t : t \in [T]\}$	$\{(V_i, V_j)\}$	$q_{i,j}+1$	$1/(d_{i,j}p_1q)$	$1 - q_{i,j}/(d_{i,j}p_1q)$
playing the role of	G	$\mathcal{U}$	$\mathcal{U}'$	s	$p_i : i < s$	p_s

Then by (L2), for each $ij \in E(R)$, we obtain edge-disjoint subgraphs $E_{i,j}^1, \dots, E_{i,j}^{q_{i,j}+1}$ of $G''[V_i, V_j]$ satisfying the following for all $t \in [T]$ and $\ell \in [q_{i,j}]$:

(E1) for each $v \in V_i$, we have $d_{E_{i,j}^\ell, V_j^t}(v) = \frac{1}{d_{i,j}p_1q} d_{G'', V_j^t}(v) \pm n^{2/3}$,

(E2) $E_{i,j}^\ell$ is $(8\varepsilon^2, 1/q)$-regular.

Recall that we have chosen a collection $\mathcal{Q} = \{Q_{1,\kappa/T}, \dots, Q_{T,\kappa/T}\}$ of edge-disjoint subgraphs of R_{multi} satisfying (Q1)–(Q5). Let $\psi : E(R_{\text{multi}}) \to \mathbb{N}$ be a function such that

$$\psi(E_{R_{\text{multi}}}(i,j)) = [q_{i,j}].$$

For all $ij \in E(R')$, there are exactly $q_{i,j}$ edges between i and j in R_{multi}, so such a function ψ exists. Now, for all $t \in [T]$, $s \in [\kappa/T]$, we let

$$G_{t,s} := \bigcup_{ij \in E(Q_{t,s})} E_{i,j}^{\psi(ij)}. \tag{6.10}$$

Since $\mathcal{Q}$ is a collection of edge-disjoint subgraphs of R_{multi} and $E_{i,j}^1, \dots, E_{i,j}^{q_{i,j}+1}$ are edge-disjoint subgraphs of G'', the graphs $G_{1,1}, \dots, G_{T,\kappa/T}$ form edge-disjoint subgraphs of G''.

We would like to use $G_{t,s} \setminus Res_t$ and Res_t to play the roles of $G[\bigcup_{i \in [r]} V_i]$ and U in Lemma 5.1, respectively. However, $E_{i,j}^\ell \setminus Res_t$ is not necessarily super-regular and the sizes of $V_i \setminus Res_t$ are not necessarily the same for all $i \in [r]$. To ensure this, we will now choose an appropriate subset $V_i^{t,s}$ of V_i which can play the role of V_i in Lemma 5.1.

For all $t \in [T]$, $i \in [r]$ and $s \in [\kappa/T]$, let

$$V_i(t,s) := V_i \setminus (Res_t \cup \bigcup_{j \in N_{Q_{t,s}}(i)} U_i(j)) \quad \text{and} \quad m := \frac{(T-1)n}{Tr} - \frac{10\varepsilon n}{r}. \tag{6.11}$$

Then by (6.6), (6.7) and (Res2), we have

$$0 \leq |V_i(t,s)| - m \leq 15\varepsilon n/r. \tag{6.12}$$

For all $t \in [T]$ and $i \in [r]$, we apply Lemma 3.18 with the following objects and parameters.

object/parameter	κ/T	1	$s \in [\kappa/T]$	$V_i \setminus Res_t$	$V_i(t,s)$	20ε	$\lvert V_i(t,s)\rvert - m$	d
playing the role of	s	r	$i \in [s]$	A	$A_{i,1}$	ε	$m_{i,1}$	$1/2$

Then we obtain sets $W_i(t,1),\dots,W_i(t,\kappa/T)$ such that $W_i(t,s)\subseteq V_i(t,s)$ with $|V_i(t,s)\setminus W_i(t,s)|=m$ and for any $v\in V_i\setminus Res_t$, we have

$$|\{s\in[\kappa/T]: v\in W_i(t,s)\}|\leq 10\varepsilon^{1/2}\kappa/T. \tag{6.13}$$

For all $t\in[T]$, $s\in[\kappa/T]$ and $i\in V(Q_{t,s})$, let $V_i^{t,s}:=V_i(t,s)\setminus W_i(t,s)$. Let

$$V_0^{t,s}:=V_0\cup\bigcup_{i\in[r]}\bigcup_{j\in N_{Q_{t,s}}(i)}(U_i(j)\setminus Res_t)\cup\bigcup_{i\in[r]}W_i(t,s)\cup\bigcup_{i\in[r]\setminus V(Q_{t,s})}(V_i\setminus Res_t).$$

Then the sets $V_0^{t,s},\{V_i^{t,s}: i\in V(Q_{t,s})\}, Res_t$ form a partition of $V(G)$, and for each $i\in V(Q_{t,s})$

$$|V_i^{t,s}| = m:=\frac{(T-1)n}{Tr}-\frac{10\varepsilon n}{r}, \text{ and} \tag{6.14}$$

$$\begin{aligned}|V_0^{t,s}| &\overset{(6.6),(6.7),(6.12)}{\leq} 6\varepsilon n+(k-1)r(5\varepsilon^2 n/r)+15\varepsilon n+(r-|V(Q_{t,s})|)n/r\\ &\overset{(Q3)}{\leq} 25\varepsilon n.\end{aligned} \tag{6.15}$$

We now further modify V_i^t into $U_i^{t,s}$ which can play the role of U_i in Lemma 5.1. For all $(t,s)\in[T]\times[\kappa/T]$ and $i\in V(Q_{t,s})$, we define

$$U_i^{t,s}:=V_i^t\setminus\bigcup_{j\in N_{Q_{t,s}}(i)}U_i(j) \text{ and } U_0^{t,s}:=\bigcup_{i\in[r]\setminus V(Q_{t,s})}V_i^t\cup\bigcup_{i\in V(Q_{t,s})}\bigcup_{j\in N_{Q_{t,s}}(i)}U_i(j).$$

Note that for each $(t,s)\in[T]\times[\kappa/T]$, the sets $\{U_0^{t,s}\}\cup\{U_i^{t,s}: i\in V(Q_{t,s})\}$ form a partition of Res_t. By (6.6), for all $(t,s)\in[T]\times[\kappa/T]$ and $i\in V(Q_{t,s})$, we have

$$|U_i^{t,s}|=|V_i^t|\pm 5k\varepsilon^2 n/r\overset{(Res2)}{=}\frac{(1\pm 8\varepsilon)n}{Tr} \text{ and } |U_0^{t,s}|\overset{(6.6)}{\leq}\sum_{i\in[r]\setminus V(Q_{t,s})}|V_i^t|+5k\varepsilon^2 n\overset{(Q3)}{\leq}2\varepsilon n. \tag{6.16}$$

Note that for all $(t,s)\in[T]\times[\kappa/T]$ and $i\in V(Q_{t,s})$, we have $U_i^{t,s},V_i^{t,s}\subseteq V_i$. Thus Proposition 3.2 with (6.14), (6.16), (L2) and the definition of p_ℓ implies that for all $(t,s)\in[T]\times[\kappa/T]$, $ij\in E(R[V(Q_{t,s})])$ and $i'j'\in E(Q_{t,s})$, we have

$$G_t^*[U_i^{t,s},U_j^{t,s}], G_t^*[V_i^{t,s},U_j^{t,s}] \text{ and } F_{t,s}[V_{i'}^{t,s},U_{j'}^{t,s}] \text{ are } (\varepsilon,(d^2))^+\text{-regular}. \tag{6.17}$$

Moreover, for all $(t,s)\in[T]\times[\kappa/T]$, $ij\in E(Q_{t,s})$ and $u\in U_i^{t,s}$, we have

$$\begin{aligned}d_{F_{t,s},V_j^{t,s}}(u) &\overset{(6.9),(6.14),(Res2)}{\geq} d_{F_t[V_i^t,V_j\setminus Res_t]}(u)-n/(Tr)\overset{(L1),(Res1)}{\geq} d\cdot d_{G',V_j}(u)-3n/(Tr)\\ &\overset{(6.5),(6.6)}{\geq} d\cdot(d_{i,j}-\varepsilon^2)|V_j|-4n/(Tr)\overset{(Res2)}{\geq}(2d^2/3)|V_j\setminus Res_t|.\end{aligned} \tag{6.18}$$

We obtain the third inequality from the definition of $U_i^{t,s}$ and the fact that $ij\in E(Q_{t,s})$.

Claim 8. *For all $t\in[T], s\in[\kappa/T]$ and $ij\in E(Q_{t,s})$, the graph $G_{t,s}[V_i^{t,s},V_j^{t,s}]$ is $(\varepsilon^{1/2},1/q)$-super-regular.*

Proof. Let $\ell\in[q_{i,j}]$ be such that $G_{t,s}[V_i,V_j]=E_{i,j}^\ell$. Such an ℓ exists by the definition of $G_{t,s}$ and the assumption that $ij\in E(Q_{t,s})$. Note that for $i'\in\{i,j\}$ we have $V_{i'}^{t,s}\subseteq V_{i'}$ with $|V_{i'}^{t,s}|=m>\frac{1}{2}|V_{i'}|$ by (6.14). Thus Proposition 3.2 with (E2) implies that $G_{t,s}[V_i^{t,s},V_j^{t,s}]=E_{i,j}^\ell[V_i^{t,s},V_j^{t,s}]$ is $(16\varepsilon^2,1/q)$-regular.

Consider $v\in V_i^{t,s}$. By the definition of $V_i^{t,s}$, we have $v\notin U_i(j)$. Thus

$$\begin{aligned}d_{G_{t,s},V_j^{t,s}}(v) &\overset{(6.6),(6.12)}{=} d_{E_{i,j}^\ell,V_j\setminus Res_t}(v)\pm\frac{16\varepsilon n}{r}=\sum_{t'\in[T]\setminus\{t\}}d_{E_{i,j}^\ell,V_j^{t'}}(v)\pm\frac{16\varepsilon n}{r}\\ &\overset{(E1)}{=}\sum_{t'\in[T]\setminus\{t\}}\frac{1}{d_{i,j}p_1 q}d_{G'',V_j^{t'}}(v)\pm\frac{17\varepsilon n}{r}\overset{(L1)}{=}\sum_{t'\in[T]\setminus\{t\}}\frac{1}{d_{i,j}q}d_{G',V_j^{t'}}(v)\pm\frac{18\varepsilon n}{r}\end{aligned}$$

$$\overset{\text{(Res1)}}{=} \frac{(T-1)}{d_{i,j}qT}d_{G',V_j}(v) \pm \frac{19\varepsilon n}{r} \overset{(6.5)}{=} \frac{(T-1)}{d_{i,j}qT}((d_{i,j}\pm\varepsilon^2)|V'_j| \pm |U'_j|) \pm \frac{19\varepsilon n}{r}$$
$$\overset{(6.6)}{=} \frac{(T-1)n}{qTr} \pm \frac{30\varepsilon n}{r} \overset{(6.14)}{=} (\frac{1}{q} \pm \varepsilon^{1/2})|V_j^{t,s}|.$$

Similarly, for $v \in V_j^{t,s}$, we have $d_{G_{t,s},V_i^{t,s}}(v) = (\frac{1}{q}\pm\varepsilon^{1/2})|V_i^{t,s}|$. Thus $G_{t,s}[V_i^{t,s}, V_j^{t,s}]$ is $(\varepsilon^{1/2}, 1/q)$-super-regular. This proves the claim. □

For all $t \in [T]$, $v \in Res_t$ and $s \in [\kappa/T]$, we know that

$$d_{G_t^*,V_i^{t,s}}(v) = d_{G_t^*,V_i}(v) \pm |V_i \setminus V_i^{t,s}| \overset{\text{(L1)}}{=} \sum_{\ell\in[T]}(d\cdot d_{G',V_i^\ell}(v) \pm n^{2/3}) \pm |V_i \setminus V_i^{t,s}|$$
$$\overset{\text{(Res1)},(6.14)}{=} d\cdot d_{G',V_i}(v) \pm 2n/(Tr).$$

This implies that

$$|\{i \in V(Q_{t,s}) : d_{G_t^*,V_i^{t,s}}(v) \geq d^2m/2\}| \geq |\{i \in V(Q_{t,s}) : d_{G',V_i}(v) \geq d|V_i|\}|$$
$$\geq \frac{d_{G'}(v) - |V_0| - dn}{\max_{i\in[r]}|V_i|} - |[r]\setminus V(Q_{t,s})| \overset{(6.7),(Q3)}{\geq} (1-1/k+\sigma/2)r. \tag{6.19}$$

We obtain the final inequality since $\delta(G') \geq (\delta - \xi - 2d)n \geq (1-1/k+3\sigma/4)n$ by (i) and (R4). This together with (6.17) and Claim 8 will ensure that $G_{t,s}\cup G_t^*$ can play the role of G in Lemma 5.1, and (6.18) shows that $F_{t,s}$ can play the role of F in Lemma 5.1.

The remaining part of this step is to construct a graph which can play the role of F' in Lemma 5.1. F' needs to contain suitable stars centred at v whenever $v \in V_0^{t,s}$. (For each t, the number of stars we will need for v in order to deal with all $s \in [\kappa/T]$ is bounded from above by (6.23).) For all $t \in [T]$, $s \in [\kappa/T]$, $v \in V(G)$ and $u \in Res_t$, let

$$I_t(v) := \{s' \in [\kappa/T] : v \in V_0^{t,s'}\} \quad \text{and} \quad i_t^s(v) := |I_t(v)\cap[s]|,$$
$$J_t(u) := \{s' \in [\kappa/T] : u \in U_0^{t,s'}\} \quad \text{and} \quad j_t^s(u) := |J_t(u)\cap[s]|. \tag{6.20}$$

Note that if $v \in V_0$, then $I_t(v) = [\kappa/T]$. If $v \in V_i\setminus Res_t$ for some $i \in [r]$, then $s \in I_t(v)$ means $v \in W_i(t,s) \cup \bigcup_{j\in N_{Q_{t,s}}(i)} U_i(j) \cup \bigcup_{i'\in[r]\setminus V(Q_{t,s})} V_{i'}$. Together with the fact that $U'_i \subseteq V_0$ and so $v \notin U'_i$, this implies

$$|I_t(v)| \overset{(Q5)}{\leq} |\{s\in[\kappa/T] : v \in W_i(t,s)\}| + |\{j \in [r] : v \in U_i(j)\}| + |\{s \in [\kappa/T] : i \notin V(Q_{t,s})\}|$$
$$\overset{(6.5),(6.13),(Q4)}{\leq} 10\varepsilon^{1/2}\kappa/T + \varepsilon r + \varepsilon r \overset{(Q2)}{\leq} 20\varepsilon^{1/2}r. \tag{6.21}$$

Similarly, for $u \in V_i^t$, we have

$$|J_t(u)| \leq |\{j \in [r] : u \in U_i(j)\}| + |\{s \in [\kappa/T] : i \notin V(Q_{t,s})\}| \overset{(6.5),(Q4)}{\leq} \varepsilon r + \varepsilon r \leq 2\varepsilon r. \tag{6.22}$$

For each $v \in V(G)\setminus Res_t$, let

$$\kappa_v := \begin{cases} (1+d)\kappa & \text{if } v \in V_0, \\ \lceil r/(2k)\rceil & \text{if } v \notin V_0. \end{cases} \tag{6.23}$$

κ_v is the overall number of stars centred at v that we will construct for given t. Note that for all $t \in [T]$ and $s \in [\kappa/T]$, no edge of $E(G'[V_0, Res_t])$ belongs to any of the graphs $G_{t,s}, G_t^*, F_t, F_t^*$. Now for each $t \in [T]$, we use these edges and edges in F_t^* to construct stars $F'_t(v,s)$ centred at v, and subsets $C_{v,s}^t$, $C_{v,s}^{*,t}$ of $[r]$ for all $v \in V(G)\setminus Res_t$ and $s \in [\kappa_v]$, in such a way that the following hold for all $t \in [T]$ and $v \in V(G)\setminus Res_t$.

(F′1) For each $s \in [\kappa_v]$, we have $C_{v,s}^t \subseteq C_{v,s}^{*,t}$, $|C_{v,s}^t| = k-1$, $|C_{v,s}^{*,t}| = k$ and $R[C_{v,s}^{*,t}] \simeq K_k$,

(F′2) for each $i \in [r]$, we have $|\{s \in [\kappa_v] : i \in C_{v,s}^{*,t}\}| \leq (k+1)q$,

(F′3) for each $s \in [\kappa_v]$, if $i \in C_{v,s}^t$, then $d_{F'_t(v,s),V_i^t}(v) \geq \frac{|V_i^t|}{q}$.

Claim 9. *For all $t \in [T]$, $v \in V(G) \setminus Res_t$ and $s \in [\kappa_v]$, there exist edge-disjoint stars $F'_t(v,s) \subseteq G'[V_0, Res_t] \cup F^*_t$ centred at v, and subsets $C^t_{v,s}$, $C^{*,t}_{v,s}$ of $[r]$ which satisfy (F′1)–(F′3).*

When applying Lemma 5.1 in Step 3 to pack $\mathcal{H}_{t,s}$, we will only make use of those stars $F'_t(v,s)$ with $v \in V_0^{t,s}$, but it is slightly more convenient to define them for all $v \in V(G)\setminus Res_t$.

Proof. First, consider $t \in [T]$ and $v \in V_0$. Then we have

$$d_{G',Res_t}(v) = \sum_{i\in[r]} d_{G',V_i^t}(v) \overset{\text{(Res1)}}{=} \frac{1}{T}\sum_{i\in[r]} d_{G',V_i}(v) \pm rn^{2/3} \overset{\substack{(i),\text{(R4)},\\ \text{(Res3)},(6.7)}}{=} (\delta \pm 3d)|Res_t|. \tag{6.24}$$

For all $v \in V_0$, $t \in [T]$ and $i \in [r]$, let $q^t_{v,i} := \lfloor \frac{q\cdot d_{G',V_i^t}(v)}{|V_i^t|} \rfloor$. Consider edge-disjoint subsets $E^t_{v,i}(1),\dots,E^t_{v,i}(q^t_{v,i})$ of $E_{G'}(\{v\},V_i^t)$ such that $|E^t_{v,i}(q')| = \frac{1}{q}|V_i^t|$ for each $q' \in [q^t_{v,i}]$. Let R^t_v be an auxiliary graph such that

$$V(R^t_v) := \{(i,q') : i \in [r], q' \in [q^t_{v,i}]\} \quad\text{and}\quad E(R^t_v) := \{(i,q')(j,q'') : ij \in E(R), q' \in [q^t_{v,i}], q'' \in [q^t_{v,j}]\}.$$

Note that each (i,q') corresponds to the star $E^t_{v,i}(q')$ centred at v. We aim to find a collection of vertex-disjoint cliques of size $k-1$ in R^t_v, which will give us edge-disjoint stars in $E_{G'}(\{v\},Res_t)$. From the definition, we have

$$|V(R^t_v)| = \sum_{i\in[r]} q^t_{v,i} \overset{\text{(Res2)}}{=} \frac{(1\pm 10\varepsilon)d_{G',Res_t}(v)q}{n/(Tr)} \pm r \overset{(6.24)}{=} \frac{(\delta \pm 4d)q|Res_t|}{n/(Tr)} \overset{\text{(Res3)}}{=} (\delta \pm 5d)qr. \tag{6.25}$$

Then, for $(i,q') \in V(R^t_v)$, we have

$$\begin{aligned} d_{R^t_v}((i,q')) &\geq q\sum_{j\in N_R(i)} d_{G',V_j^t}(v)|V_j^t|^{-1} - d_R(i) \overset{\text{(Res2)}}{\geq} \frac{Tqr}{(1+7\varepsilon)n}\sum_{j\in N_R(i)} d_{G',V_j^t}(v) - r \\ &\geq \frac{Tqr}{(1+7\varepsilon)n}\Big(\sum_{j\in N_R(i)} |V_j^t| - \sum_{j\in[r]}(|V_j^t| - d_{G',V_j^t}(v))\Big) - r \\ &\overset{\substack{(6.4),(6.24),\\ \text{(Res2)},\text{(Res3)}}}{\geq} (2\delta - 2d^{1/2} - 1)qr - r \overset{(6.25)}{\geq} (1 - \frac{1}{k-1} + \sigma)|V(R^t_v)|. \end{aligned} \tag{6.26}$$

Here, the final inequality follows from (3.6). By the Hajnal-Szemerédi theorem, R^t_v contains at least

$$|V(R^t_v)|/(k-1) - 1 \overset{(6.25)}{\geq} (\delta - 5d)qr/(k-1) - 1 \overset{\text{(Q2)}}{\geq} (1+d)\kappa \overset{(6.23)}{=} \kappa_v$$

vertex-disjoint copies of K_{k-1}. Let $C^t_v(1),\dots,C^t_v(\kappa_v)$ be such vertex-disjoint copies of K_{k-1} in R^t_v. For each $s \in [\kappa_v]$, we let

$$F'_t(v,s) := \bigcup_{(i,q')\in V(C^t_v(s))} E^t_{v,i}(q') \qquad\text{and}\qquad C^t_{v,s} := \{i : (i,q') \in V(C^t_v(s)) \text{ for some } q' \in [q^t_{v,i}]\}.$$

By construction $|C^t_{v,s}| = k-1$ and $R[C^t_{v,s}] \simeq K_{k-1}$. Moreover, the maximum degree of the multi-$(k-1)$-graph $\{C^t_{v,s} : s \in [\kappa_v]\}$ is at most q. Thus we can apply Lemma 3.22 with $\{C^t_{v,s} : s \in [\kappa_v]\}$, R, q and k playing the roles of $\mathcal{F}, R, q$ and k. Then we obtain sets $C^{*,t}_{v,s}$ satisfying the following for all $s \in [\kappa_v]$ and $i \in [r]$:

$$C^t_{v,s} \subseteq C^{*,t}_{v,s},\quad R[C^{*,t}_{v,s}] \simeq K_k,\quad\text{and}\quad |\{s \in [\kappa_v] : i \in C^{*,t}_{v,s}\}| \leq (k+1)q. \tag{6.27}$$

It is easy to see that for all $s \in [\kappa_v]$ the sets $C^t_{v,s}$, $C^{*,t}_{v,s}$ and the stars $F'_t(v,s)$ satisfy (F′1)–(F′3).

Now, we consider $t \in [T]$ and $v \in V_i \setminus Res_t$ with $i \in [r]$. Let $S^t_v := N_R(i) \setminus \{j : v \in U_i(j)\}$, and for each $j \in S^t_v$, let $E^t_{v,j}$ be a subset of $E_{F^*_t}(\{v\},V_j^t)$ with $|E^t_{v,j}| = \frac{1}{q}|V_j^t|$. We can choose such a star as there exists $\ell \in \{6,7\}$ such that

$$d_{F^*_t,V_j^t}(v) = d_{L_\ell,V_j^t}(v) \overset{\text{(L1)}}{=} d\cdot d_{G',V_j^t}(v) \pm n^{2/3} \overset{\text{(Res1)},\text{(Res2)}}{=} (1\pm 10\varepsilon)d\cdot d_{i,j}|V_j^t| > \frac{1}{q}|V_j^t|.$$

Here, the third equality follows since $v \notin U_i(j)$. By (6.4), (6.5) and the fact that $v \notin U'_i$, we have $|S^t_v| \geq (\delta - 2d^{1/2})r$. Thus

$$\delta(R[S^t_v]) \geq |S^t_v| - (r - \delta(R)) \overset{(6.4)}{\geq} (1 - \frac{1}{k-1})|S^t_v|.$$

Again, by the Hajnal-Szemerédi theorem, $R[S^t_v]$ contains (at least) $\kappa_v = \lceil r/(2k) \rceil$ vertex-disjoint copies of K_{k-1}. Denote their vertex sets by $C^t_{v,1}, \dots, C^t_{v,\kappa_v}$. We apply Lemma 3.22 with $\{C^t_{v,s} : s \in [\kappa_v]\}$, R, 1 and k playing the roles of $\mathcal{F}, R, q$ and k respectively, to extend each $C^t_{v,s}$ into a $C^{*,t}_{v,s}$ with $R[C^{*,t}_{v,s}] \simeq K_k$ and such that $|\{s \in [\kappa_v] : i \in C^{*,t}_{v,s}\}| \leq k+1$ for each $i \in [r]$. For each $s \in [\kappa_v]$, let $F'_t(v,s) := \bigcup_{j \in C^t_{v,s}} E^t_{v,j}$. Again, it is easy to see that for all $s \in [\kappa_v]$ the sets $C^t_{v,s}$, $C^{*,t}_{v,s}$ and the stars $F'_t(v,s)$ satisfy (F′1)–(F′3). This proves the claim. □

Altogether we will apply Lemma 5.1 κ times in Step 3. In each application, we want the leaves of the stars that we use to be evenly distributed (see condition (A8)$_{5.1}$). This will be ensured by Claim 13. More precisely, for each $v \in V(G) \setminus Res_t$, our aim is to choose a permutation $\pi^t_v : [\kappa_v] \to [\kappa_v]$ satisfying the following.

(F′4) For all $t \in [T]$, $i \in [r]$ and $s \in [\kappa/T]$, we have $C(t,s,i) \leq \varepsilon^{4/5}n/r$, where $C(t,s,i) := |\{v \in V^{t,s}_0 : i \in C^{*,t}_{v,\pi^t_v(s')}$ for some s' with $(i^s_t(v)-1)T+1 \leq s' \leq i^s_t(v)T\}|$,

(F′5) for all $t \in [T]$, $s \in [\kappa/T]$ and $t' \in [T]$, we have that $\bigcup_{v \in V^{t,s}_0} C^{*,t}_{v,\pi^t_v((i^s_t(v)-1)T+t')} \subseteq V(Q_{t,s})$.

Recall from (6.20) that $i^s_t(v)$ counts the number of $s' \in [s]$ for which $v \in V^{t,s'}_0$. The number $C(t,s,i)$ is well-defined because $i^s_t(v) \leq \kappa_v/T$ for all $v \in V(G) \setminus Res_t$ by (6.21).

Claim 10. *For each $t \in [T]$ and each $v \in V(G) \setminus Res_t$, there exists a permutation $\pi^t_v : [\kappa_v] \to [\kappa_v]$ satisfying (F′4)–(F′5).*

Proof. We fix $t \in [T]$. We claim that for each $s \in [\kappa/T] \cup \{0\}$ the following hold. For each $v \in V(G) \setminus Res_t$, there exists an injective map $\pi^t_{v,s} : [i^s_t(v)T] \to [\kappa_v]$ satisfying the following.

(F′4)^t_s For all $i \in [r]$ and $\ell \in [s]$, we have

$$|\{v \in V^{t,\ell}_0 : i \in C^{*,t}_{v,\pi^t_{v,s}(s')} \text{ for some } s' \text{ with } (i^\ell_t(v)-1)T+1 \leq s' \leq i^\ell_t(v)T\}| \leq \varepsilon^{4/5}n/r,$$

(F′5)^t_s for all $\ell \in [s]$ and $t' \in [T]$, we have that $\bigcup_{v \in V^{t,\ell}_0} C^{*,t}_{v,\pi^t_{v,s}((i^\ell_t(v)-1)T+t')} \subseteq V(Q_{t,\ell})$.

Note that both (F′4)$^t_0$ and (F′5)$^t_0$ hold by letting $\pi^t_{v,0} : \emptyset \to \emptyset$ be the empty map for all $v \in V(G) \setminus Res_t$. Assume that for some $s \in [\kappa/T - 1] \cup \{0\}$ we have already constructed injective maps $\pi^t_{v,s}$ for all $v \in V(G) \setminus Res_t$ which satisfy (F′4)^t_s and (F′5)^t_s. For each $v \in V^{t,s+1}_0$, we consider the set

$$A_v := \{s' \in [\kappa_v] \setminus \pi^t_{v,s}([i^s_t(v)T]) : C^{*,t}_{v,s'} \subseteq V(Q_{t,s+1})\}.$$

Then we have

$$\begin{aligned} |A_v| &\overset{(F'2)}{\geq} \kappa_v - i^s_t(v)T - (k+1)q(r - |V(Q_{t,s+1})|) \\ &\overset{(6.21),(Q3)}{\geq} \min\{d \cdot \kappa, r/(2k) - 20T\varepsilon^{1/2}r\} - (k+1)q\varepsilon r \geq r/(4k). \end{aligned} \tag{6.28}$$

We choose a subset $I_v \subseteq A_v$ of size T uniformly at random. Then (F′2) implies that for each $i \in V(Q_{t,s+1})$ we have

$$\mathbb{P}[i \in \bigcup_{s' \in I_v} C^{*,t}_{v,s'}] \leq (k+1)qT/|A_v| \leq 10qk^2T/r.$$

Thus

$$\mathbb{E}[|\{v \in V^{t,s+1}_0 : i \in \bigcup_{s' \in I_v} C^{*,t}_{v,s'}\}|] \leq 10qk^2T|V^{t,s+1}_0|/r \overset{(6.15)}{\leq} \varepsilon^{4/5}n/(2r).$$

A Chernoff bound (Lemma 3.1) gives us that for each $i \in V(Q_{t,s+1})$

$$\mathbb{P}\Big[|\{v \in V_0^{t,s+1} : i \in \bigcup_{s' \in I_v} C^{*,t}_{v,s'}\}| \geq \varepsilon^{4/5} n/r\Big] \leq \exp(-\frac{(\varepsilon^{4/5}n/(2r))^2}{2|V_0^{t,s+1}|}) \overset{(6.15)}{\leq} e^{-n/r^3}.$$

Since $1 - |V(Q_{t,s+1})|e^{-n/r^3} > 0$, the union bound implies that there exists a choice of I_v for each $v \in V_0^{t,s+1}$ such that for all $i \in V(Q_{t,s+1})$, we have that

$$|\{v \in V_0^{t,s+1} : i \in \bigcup_{s' \in I_v} C^{*,t}_{v,s'}\}| \leq \varepsilon^{4/5} n/r. \tag{6.29}$$

If $v \in V(G) \setminus (Res_t \cup V_0^{t,s+1})$ (and thus $i_t^{s+1}(v) = i_t^s(v)$), we let $\pi^t_{v,s+1} := \pi^t_{v,s}$. For each $v \in V_0^{t,s+1}$, we extend $\pi^t_{v,s}$ into $\pi^t_{v,s+1}$ by defining $\pi^t_{v,s+1} : [i_t^{s+1}(v)T] \setminus [i_t^s(v)T] \to I_v$ in an arbitrary injective way. Then, by the choice of I_v, we have that $\pi^t_{v,s+1}$ is an injective map from $[i_t^{s+1}(v)T]$ to $[\kappa_v]$ satisfying $(\mathrm{F}'5)^t_{s+1}$. Moreover, (6.29) implies that for any $i \in V(Q_{t,s+1})$, we have

$$\begin{aligned}&|\{v \in V_0^{t,s+1} : i \in C^{*,t}_{v,\pi^t_{v,s+1}(s')} \text{ for some } s' \text{ with } (i_t^{s+1}(v)-1)T+1 \leq s' \leq i_t^{s+1}(v)T\}| \\ &= |\{v \in V_0^{t,s+1} : i \in \bigcup_{s' \in I_v} C^{*,t}_{v,s'}\}| \overset{(6.29)}{\leq} \varepsilon^{4/5} n/r.\end{aligned}$$

This with $(\mathrm{F}'4)^t_s$ implies $(\mathrm{F}'4)^t_{s+1}$. By repeating this, we obtain injective maps $\pi^t_{v,\kappa/T}$ satisfying both $(\mathrm{F}'4)^t_{\kappa/T}$ and $(\mathrm{F}'5)^t_{\kappa/T}$. For each $v \in V(G) \setminus Res_t$, we extend $\pi^t_{v,\kappa/T}$ into a permutation $\pi^t_v : [\kappa_v] \to [\kappa_v]$ by assigning arbitrary values for the remaining values in the domain. It is easy to see that $(\mathrm{F}'4)^t_{\kappa/T}$ implies (F′4) and $(\mathrm{F}'5)^t_{\kappa/T}$ implies (F′5). We can find such permutations for all $t \in [T]$. Thus such collection satisfies both (F′4) and (F′5). □

For each $t \in [T]$, let

$$G_t := G_t^* \cup \bigcup_{s \in [\kappa/T]} G_{t,s} \text{ and } F'_t := \bigcup_{v \in V(G) \setminus Res_t} \bigcup_{s \in [\kappa_v]} F'_t(v,s).$$

Then $G_1, \dots, G_T, F_1, \dots, F_T, F'_1, \dots, F'_T$ form edge-disjoint subgraphs of G. (Recall that G_t^* was defined in (6.8), $G_{t,s}$ in (6.10) and $F'_v(t,s)$ in Claim 9.)

Step 2. Partitioning $\mathcal{H}$. Now we will partition $\mathcal{H}$. Recall that the graphs in $\mathcal{H}$ are η^2-separable. By packing several graphs from $\mathcal{H}$ with less than $n/4$ edges suitably into a single graph in a way that no edges from distinct graphs intersect each other, we can assume that all but at most one graph in $\mathcal{H}$ have at least $n/4$ edges, and that all graphs in $\mathcal{H}$ are (k,η)-chromatic, η-separable and have maximum degree at most Δ. By adding at most $n/4$ edges to at most one graph if necessary, we can then assume that all graphs in $\mathcal{H}$ have at least $n/4$ edges. Moreover, if $e(\mathcal{H})$ is too small, we can add some copies of n-vertex paths to $\mathcal{H}$ to assume that

$$\varepsilon n^2 \leq e(\mathcal{H}) \overset{(iii)}{\leq} (1-\nu)e(G) + n/4.$$

We partition $\mathcal{H}$ into κ collections $\mathcal{H}_{1,1}, \dots, \mathcal{H}_{T,\kappa/T}$ such that for all $t \in [T]$ and $s \in [\kappa/T]$, we have

$$n^{7/4} \overset{(\mathrm{Q2})}{\leq} \frac{\varepsilon n^2}{\kappa} - \Delta n \leq e(\mathcal{H}_{t,s}) < \frac{1}{\kappa}(1-\nu)e(G) + 2\Delta n \overset{(i),(\mathrm{Q2})}{\leq} \frac{(1-2\nu/3)(k-1)n^2}{2qr}. \tag{6.30}$$

Indeed, this is possible since $e(H) \leq \Delta n$ for all $H \in \mathcal{H}$. Now, we are ready to construct the desired packing.

Step 3. Construction of packings into the host graphs. As $G_1, \dots, G_T, F_1, \dots, F_T, F'_1, \dots, F'_T$ are edge-disjoint subgraphs of G, and $\mathcal{H}_{1,1}, \dots, \mathcal{H}_{T,\kappa/T}$ is a partition of $\mathcal{H}$, it suffices to show that for each $t \in [T]$, we can pack $\mathcal{H}_t := \bigcup_{s=1}^{\kappa/T} \mathcal{H}_{t,s}$ into $G_t \cup \bigcup_{s \in [\kappa/T]} F_{t,s} \cup F'_t$. (Recall from (6.9) that $F_{t,1},\dots,F_{t,\kappa/T}$ are edge-disjoint subgraphs of F_t.) We fix $t\in[T]$ and will apply Lemma 5.1 κ/T times to show that such a packing exists.

Assume that for some s with $0\le s\le \kappa/T-1$, we have already defined a function ϕ_s packing $\bigcup_{s'=1}^{s}\mathcal{H}_{t,s'}$ into $G_t\cup F_t\cup F'_t$ and satisfying the following, where $\Phi^s:=\bigcup_{s'=1}^{s}\phi_s(\mathcal{H}_{t,s'})$ and $j_t^s(u)$ is defined in (6.20) and G_t^* is defined in (6.8).

(G1)$_s$ For each $u\in Res_t$, we have $d_{\Phi^s\cap G_t^*}(u)\le \frac{4k\Delta j_t^s(u)n}{qr}+\frac{\varepsilon^{1/9}sn}{r}$,
(G2)$_s$ for each $i\in[r]$, we have $e_{\Phi^s\cap G_t^*}(V_i\backslash V_i^t, Res_t)\le \frac{\varepsilon^{1/3}sn^2}{r^2}$,
(G3)$_s$ for $s'\in[\kappa/T]\setminus[s]$, we have $E(\Phi^s)\cap(E(G_{t,s'})\cup E(F_{t,s'}))=\emptyset$,
(G4)$_s$ for $v\in V(G)\setminus Res_t$, $s''\in[\kappa_v]$ with $s''>i_t^s(v)\cdot T$, we have $E(\Phi^s)\cap F'_t(v,\pi_v^t(s''))=\emptyset$.

Note that (G1)$_0$–(G4)$_0$ trivially hold with an empty packing $\phi_0:\emptyset\to\emptyset$. For each $t'\in[T]$ and $v\in V(G)\setminus Res_t$, let $\ell(v,t'):=\pi_v^t((i_t^{s+1}(v)-1)T+t')$. (Note that $\ell(v,t')$ is well-defined since $(i_t^{s+1}(v)-1)T+t'\le\kappa_v$ by (6.21).) Let

$$V:=\bigcup_{i\in V(Q_{t,s+1})}V_i^{t,s+1},\quad U:=\bigcup_{i\in V(Q_{t,s+1})}U_i^{t,s+1}, \tag{6.31}$$

$$\hat{G}:=(G_{t,s+1}[V]\cup G_t^*[V\cup Res_t])\setminus E(\Phi^s),\ \text{ and }\ \hat{F}':=\bigcup_{v\in V_0^{t,s+1}}\bigcup_{t'\in[T]}F'_t(v,\ell(v,t'))[\{v\},U]. \tag{6.32}$$

Note that (G3)$_s$ implies that $E(F_{t,s+1})\cap E(\Phi^s)=\emptyset$. Let $\hat{R}$ be the graph on vertex set $V(Q_{t,s+1})$ with

$$E(\hat{R}):=\{ij\in E(R[V(Q_{t,s+1})]):|E_{G_t^*}(V_i,V_j)\cap E(\Phi^s)|<\varepsilon^{1/10}n^2/r^2\}.$$

We wish to apply Lemma 5.1 with the following objects and parameters.

object/parameter	$\hat{G}$	$F_{t,s+1}[V,U]$	$\hat{F}'$	$V_0^{t,s+1}$	$U_0^{t,s+1}$	$V_i^{t,s+1}$	$U_i^{t,s+1}$	$\hat{R}$
playing the role of	G	F	F'	V_0	U_0	V_i	U_i	R
object/parameter	$1/q$	$\mathcal{H}_{t,s+1}$	d	$C^{*,t}_{v,\ell(v,t')}$	$C^t_{v,\ell(v,t')}$	$F'_t(v,\ell(v,t'))[\{v\},U]$	k	Δ
playing the role of	α	$\mathcal{H}$	d	$C^*_{v,t}$	$C_{v,t}$	$F'_{v,t}$	k	Δ
object/parameter	$Q_{t,s+1}$	η	25ε	$\sigma/2$	T	$\nu/2$	m	
playing the role of	Q	η	ε	σ	T	ν	n'	

Thus $Res_t\setminus U_0^{t,s+1}$ plays the role of $U=\bigcup_{i=1}^r U_i$ in Lemma 5.1, and $t'\in[T]$ stands for $t\in[T]$. By (6.1), (6.14), (6.15), (6.16), (Q3) and (F′5) we have appropriate objects and parameters as well as the hierarchy of constants required in Lemma 5.1. Now we show that (A1)$_{5.1}$–(A9)$_{5.1}$ hold. (A1)$_{5.1}$ is obvious from Theorem 1.2 (ii) and our assumption in Step 2. (A2)$_{5.1}$ holds by (6.30). (A3)$_{5.1}$ follows from Claim 8 and (G3)$_s$. Consider $ij\in E(\hat{R})$, then $\hat{G}[U_i^{t,s+1},U_j^{t,s+1}]=G_t^*[U_i^{t,s+1},U_j^{t,s+1}]\setminus E(\Phi^s)$. Since $U_i^{t,s+1}\subseteq V_i$ and $U_j^{t,s+1}\subseteq V_j$, the properties (6.16), (6.17) and the definition of $\hat{R}$ imply that

$$e_{G_t^*}(U_i^{t,s+1},U_j^{t,s+1})-e_{\Phi^s\cap G_t^*}(V_i,V_j)\ge(1-\varepsilon^{1/15})e_{G_t^*}(U_i^{t,s+1},U_j^{t,s+1}).$$

Thus, Proposition 3.3 with (6.17) implies that $\hat{G}[U_i^{t,s+1},U_j^{t,s+1}]$ is $(\varepsilon^{1/50},(d^2))^+$-regular. The calculation for $\hat{G}[V_i^{t,s+1},U_j^{t,s+1}]$ is similar. Thus (A4)$_{5.1}$ holds with the above objects and parameters. By (G1)$_s$, for each $i\in[r]$ we have

$$e_{\phi^s\cap G_t^*}(V_i^t,\bigcup_{j\in[r]\setminus\{i\}}V_j)\le\sum_{v\in V_i^t}\left(\frac{4k\Delta j_t^s(v)n}{qr}+\frac{\varepsilon^{1/9}sn}{r}\right)\overset{\text{(Q2)},(6.22),\text{(Res2)}}{\le}\frac{\varepsilon^{1/9}n^2}{r}. \tag{6.33}$$

Thus, for $i\in V(Q_{t,s+1})=V(\hat{R})$, we have

$$\begin{aligned} d_R(i)-d_{\hat{R}}(i)&\le\frac{e_{\Phi^s\cap G_t^*}(V_i\backslash V_i^t,Res_t)+e_{\Phi^s\cap G_t^*}(V_i^t,\bigcup_{j\in[r]\setminus\{i\}}V_j)}{\varepsilon^{1/10}n^2/r^2}+|V(R)\setminus V(\hat{R})|\\ &\overset{\text{(G2)}_s,\text{(Q3)},(6.33)}{\le}\frac{\varepsilon^{1/3}sn^2/r^2+\varepsilon^{1/9}n^2/r}{\varepsilon^{1/10}n^2/r^2}+\varepsilon r\overset{\text{(Q2)}}{\le}\varepsilon^{1/100}r.\end{aligned}$$

This with (6.4) and (3.6) implies that (A5)$_{5.1}$ holds for $\hat{R}$. For all $ij \in E(Q_{t,s+1})$ and $u \in U_i^{t,s+1}$, by (6.18), we have

$$d_{F_{t,s+1},V_j^{t,s+1}}(u) \geq 2d^2|V_j \setminus Res_t|/3 \overset{(\text{Res2}),(6.14)}{\geq} d^3m.$$

Thus (A6)$_{5.1}$ holds. By (F′1), (F′4) and the fact that $i_t^{s+1}(v) = i_t^s(v)+1$ for all $v \in V_0^{t,s}$, (A8)$_{5.1}$ holds (for $C^{*,t}_{v,\ell(v,t')}$, $C^t_{v,\ell(v,t')}$ and all $v \in V_0^{t,s}$). If $v \in V_0^{t,s}, t' \in [T]$ and $i \in C^t_{v,\ell(v,t')} \subseteq C^{*,t}_{v,\ell(v,t')}$ then (F′5) implies that $i \in V(Q_{t,s+1})$. Moreover, by (6.16) we have $|U_i^{t,s+1}| \geq |V_i^t| - 5k\varepsilon^2 n/r$. Together with (F′3) this implies that $d_{F'_{t'}(v,\ell(v,t')),U_i^{t,s+1}}(v) \geq (1-\varepsilon)|U_i^{t,s+1}|/q$. Thus (A7)$_{5.1}$ holds. To check (A9)$_{5.1}$, note that for each $u \in U_0^{t,s+1}$, we have

$$d_{G_t^* \cap \Phi^s}(u) \overset{(\text{G1})_s}{\leq} 4k\Delta j_t^s(u)n/(qr) + \varepsilon^{1/9}sn/r \overset{(\text{Q2}),(6.22)}{\leq} \varepsilon^{1/10}n.$$

Thus,

$$\begin{aligned}|\{i \in V(Q_{t,s+1}) : d_{\hat{G},V_i^{t,s+1}}(u) \geq d^2m/3\}| &\geq |\{i \in V(Q_{t,s+1}) : d_{G_t^*,V_i^{t,s+1}}(u) \geq d^2m/2\}| - \frac{d_{G_t^*\cap\Phi^s}(u)}{d^2m/6}\\ &\overset{(6.19)}{\geq} (1-1/k+\sigma/2)r - \frac{\varepsilon^{1/10}n}{d^2m/6} \overset{(6.14)}{\geq} (1-1/k+\sigma/3)r.\end{aligned}$$

This implies that

$$|\{i \in V(Q_{t,s+1}) : d_{\hat{G},V_j^{t,s+1}}(u) \geq d^3m \text{ for all } j \in N_{Q_{t,s+1}}(i)\}| \geq \sigma^2 r.$$

This shows that (A9)$_{5.1}$ holds. Hence, by Lemma 5.1, we obtain a function ψ_{s+1} packing $\mathcal{H}_{t,s+1}$ into $\hat{G} \cup F_{t,s+1} \cup \hat{F}'$ and satisfying the following.

- (B1) $\Delta(\psi_{s+1}(\mathcal{H}_{t,s+1})) \leq 4k\Delta n/(qr)$,
- (B2) for each $u \in Res_t \setminus U_0^{t,s+1}$, we have $d_{\psi_{s+1}(\mathcal{H}_{t,s+1})\cap\hat{G}}(u) \leq 10\Delta\varepsilon^{1/8}n/r$,
- (B3) for each $i \in V(Q_{t,s})$, we have $e_{\psi_{s+1}(\mathcal{H}_{t,s+1})\cap\hat{G}}(V_i^{t,s+1}, Res_t) < 10\varepsilon^{1/2}n^2/r^2$.

Moreover, (G3)$_s$ with (G4)$_s$ implies that $\psi_{s+1}(\mathcal{H}_{t,s+1})$ is edge-disjoint from Φ^s, thus the map $\phi_{s+1} := \phi_s \cup \psi_{s+1}$ packs $\bigcup_{s'=1}^{s+1}\mathcal{H}_{t,s'}$ into $G_t \cup \bigcup_{s'=1}^{\kappa/T} F_{t,s'} \cup F'_t$. Now it remains to show that ϕ_{s+1} satisfies (G1)$_{s+1}$–(G4)$_{s+1}$.

Consider any vertex $u \in Res_t$. If $u \in U_0^{t,s+1}$, then we know that $j_t^{s+1}(u) = j_t^s(u)+1$. Thus (G1)$_s$ together with (B1) implies (G1)$_{s+1}$ for the vertex u. If $u \in Res_t \setminus U_0^{t,s+1}$, then we have $j_t^{s+1}(u) = j_t^s(u)$, thus (G1)$_s$ together with (B2) implies (G1)$_{s+1}$.

For each $i \in [r]$, (6.31) implies that the vertices in $V_i \setminus (V_i^{t,s+1} \cup V_i^t) \subseteq V_0^{t,s+1}$ are not incident to any edges in $\Phi^{s+1} \cap G_t^*$. Thus it is easy to see that (G2)$_s$ together with (B3) implies (G2)$_{s+1}$. As ψ_{s+1} packs $\mathcal{H}_{t,s+1}$ into $\hat{G} \cup F_{t,s+1} \cup \hat{F}'$, (6.32) together with (G3)$_s$ implies (G3)$_{s+1}$. Moreover, we have

$$i_t^{s+1}(v) = \begin{cases} i_t^s(v)+1 & \text{if } v \in V_0^{t,s+1},\\ i_t^s(v) & \text{otherwise.}\end{cases}$$

Thus, (6.32) together with (G4)$_s$ and the definition of $\ell(v,t')$ implies (G4)$_{s+1}$.

By repeating this for each $s \in [\kappa/T]$ in order, we obtain a function $\phi_{\kappa/T}$ which packs $\mathcal{H}_t$ into $G_t \cup F_t \cup F'_t$. By taking the union of such functions over all $t \in [T]$, we obtain a desired function packing $\mathcal{H}$ into $\bigcup_{t\in[T]} G_t \cup F_t \cup F'_t \subseteq G$. This completes the proof. □

The proof of Theorem 1.5, follows almost exactly the same lines as that of Theorem 1.2, with one very minor difference. Indeed, the only place where we need the condition that G is almost regular is when we apply Lemma 3.13 in Step 1 to obtain (Q1)–(Q5). Thus to prove Theorem 1.5, we only need to replace the application of Lemma 3.13 with an application of the following result. (Note that (B1) below implies both (Q3) and (Q4).)

Lemma 6.1. *Suppose $n, q, T \in \mathbb{N}$ with $0 < 1/n \ll \varepsilon, 1/T, 1/q, \nu \leq 1/2$ and $0 < 1/n \ll \nu < \sigma/2 < 1$ and $\delta = 1/2+\sigma$ and q divides T. Let G be an n-vertex multi-graph with edge-multiplicity at most q, such that for all $v \in V(G)$ we have $d_G(v) \geq q\delta n$.*

Then there exists a subset $V' \subseteq V(G)$ with $|V'| \leq 1$ and $|V(G)\backslash V'|$ being even, and there exist pairwise edge-disjoint matchings $F_{1,1}, \dots, F_{1,\kappa}, F_{2,1}, \dots, F_{T,\kappa}$ of G with $\kappa = \frac{(\delta+\sqrt{2\delta-1}-\nu)qn}{2T} \pm 1$ satisfying the following.

(B1) *For each $(t', i) \in [T] \times [\kappa]$, we have that $V(F_{t',i}) = V(G)\backslash V'$,*

(B2) *for all $t' \in [T]$ and $u, v \in V(G)$, we have $|\{i \in [\kappa] : u \in N_{F_{t',i}}(v)\}| \leq 1$.*

The proof of the above lemma is very similar (but simpler) than that of Lemma 3.13. We proceed as in the proof of Lemma 3.13 to obtain simple graphs G^c with $\delta(G^c) > \delta n - \nu^2 n$. We let $V' \subseteq V(G)$ be such that $|V'| \leq 1$ and $|V(G)\backslash V'|$ is even. The difference is that we now apply the following result of [11] to each $G^c_* := G^c[V(G)\backslash V']$ to obtain the desired matchings M^c_i: for every $\alpha > 0$, any sufficiently large n-vertex graph with minimum degree $\delta \geq (1/2+\alpha)n$ contains at least $(\delta - \alpha n + \sqrt{n(2\delta - n)})/4$ edge-disjoint Hamilton cycles.

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